

## SINGULAR POLYNOMIALS FOR THE SYMMETRIC GROUPS

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ABSTRACT. For certain negative rational numbers  $\kappa_0$ , called singular values, and associated with the symmetric group  $S_N$  on  $N$  objects, there exist homogeneous polynomials annihilated by each Dunkl operator when the parameter  $\kappa = \kappa_0$ . It was shown by the author, de Jeu and Opdam (Trans. Amer. Math. Soc. 346 (1994), 237-256) that the singular values are exactly the values  $-\frac{m}{n}$  with  $2 \leq n \leq N$ ,  $m = 1, 2, \dots$  and  $\frac{m}{n}$  is not an integer. This paper constructs for each pair  $(m, n)$  satisfying these conditions an irreducible  $S_N$ -module of singular polynomials for the singular value  $-\frac{m}{n}$ . The module is of isotype  $(n-1, (n_1-1)^l, \rho)$  where  $n_1 = n/\gcd(m, n)$ ,  $\rho = N - (n-1) - l(n_1-1)$  and  $1 \leq \rho \leq n_1-1$ . The singular polynomials are special cases of nonsymmetric Jack polynomials. The paper presents some formulae for the action of Dunkl operators on these polynomials valid in general, and a method for showing the dependence of poles (in the parameter  $\kappa$ ) on the number of variables. Murphy elements are used to analyze the representation of  $S_N$  on irreducible spaces of singular polynomials.

## 1. INTRODUCTION

We will construct polynomials on  $\mathbb{R}^N$  which are annihilated by each Dunkl operator associated with the symmetric group  $S_N$ , acting by permutation of coordinates, when the parameter takes on a singular value  $-\frac{m}{n}$  with  $2 \leq n \leq N$  and  $-\frac{m}{n} \notin \mathbb{Z}$ . The group  $S_N$  is considered as the finite reflection group of type  $A_{N-1}$ . Let  $\mathbb{N}_0$  denote  $\{0, 1, 2, 3, \dots\}$ ; for  $\alpha \in \mathbb{N}_0^N$  (called a “composition”) let  $|\alpha| = \sum_{i=1}^N \alpha_i$  and define the monomial  $x^\alpha$  to be  $\prod_{i=1}^N x_i^{\alpha_i}$ ; its degree is  $|\alpha|$ . The length of a composition is  $\ell(\alpha) = \max\{j : \alpha_j > 0\}$ . For  $1 \leq i \leq N$  let  $\varepsilon(i) \in \mathbb{N}_0^N$  denote the standard basis element, that is,  $\varepsilon(i)_j = \delta_{ij}$ . Consider elements of  $S_N$  as functions on  $\{1, 2, \dots, N\}$  then for  $x \in \mathbb{R}^N$  and  $w \in S_N$  let  $(xw)_i = x_{w(i)}$  for  $1 \leq i \leq N$ ; and extend this action to polynomials by  $wf(x) = f(xw)$ . This has the effect that monomials transform to monomials,  $w(x^\alpha) = x^{w\alpha}$  where  $(w\alpha)_i = \alpha_{w^{-1}(i)}$  for  $\alpha \in \mathbb{N}_0^N$ . (Consider  $x$  as a row vector,  $\alpha$  as a column vector, and  $w$  as a permutation matrix, with 1’s at the  $(w(j), j)$  entries.) The reflections in  $S_N$  are the transpositions, denoted by  $(i, j)$  for  $i \neq j$ , interchanging  $x_i$  and  $x_j$ .

In [4] the author constructed for each finite reflection group a parametrized commutative algebra of differential-difference operators. Let  $\kappa$  be a formal parameter,

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*Date:* March 16, 2004.

*2000 Mathematics Subject Classification.* Primary 20C30, 05E10; Secondary 16S32.

*Key words and phrases.* singular polynomials, nonsymmetric Jack polynomials, Dunkl operators.

During the preparation of this paper the author was partially supported by NSF grant DMS 0100539.

that is,  $\mathbb{Q}(\kappa)$  is a transcendental extension of  $\mathbb{Q}$ . For the symmetric group the operators are defined as follows:

**Definition 1.** For any polynomial  $f$  on  $\mathbb{R}^N$  and  $1 \leq i \leq N$  let

$$\mathcal{D}_i f(x) = \frac{\partial}{\partial x_i} f(x) + \kappa \sum_{j \neq i} \frac{f(x) - (ij)f(x)}{x_i - x_j}.$$

The polynomials under consideration are elements of  $\text{span}_{\mathbb{Q}(\kappa)} \{x^\alpha : \alpha \in \mathbb{N}_0^N\}$ . It was shown in [4] that  $\mathcal{D}_i \mathcal{D}_j = \mathcal{D}_j \mathcal{D}_i$  for  $1 \leq i, j \leq N$  and each  $\mathcal{D}_i$  maps homogeneous polynomials to homogeneous polynomials. A specific numerical parameter value  $\kappa_0$  is said to be a *singular value* (associated with  $S_N$ ) if there exists a nonzero polynomial  $p$  such that  $\mathcal{D}_i p = 0$  for  $1 \leq i \leq N$  when  $\kappa$  is specialized to  $\kappa_0$ , and  $p$  is called a *singular polynomial*. It was shown in [7] that the singular values are the numbers  $-\frac{j}{n}$  where  $n = 2, \dots, N$ ,  $j \in \mathbb{N}$  and  $\frac{j}{n} \notin \mathbb{Z}$ . The space of homogeneous polynomials of degree  $n$ , denoted by  $\mathcal{P}_n$ , is  $\text{span}_{\mathbb{Q}(\kappa)} \{x^\alpha : \alpha \in \mathbb{N}_0^N, |\alpha| = n\}$ . The set of partitions of length  $\leq N$  is denoted by  $\mathbb{N}_0^{N,P}$  and consists of all  $\lambda \in \mathbb{N}_0^N$  such that  $\lambda_i \geq \lambda_{i+1}$  for  $1 \leq i \leq N-1$ . When writing partitions it is customary to suppress trailing zeros and to use exponents to indicate multiplicity, for example  $(5, 2^3)$  is the same as  $(5, 2, 2, 2, 0) \in \mathbb{N}_0^{5,P}$ . The irreducible representations of  $S_N$  are labeled by partitions of  $N$  (that is,  $\tau \in \mathbb{N}_0^{N,P}$  and  $|\tau| = N$ ) and we say a polynomial  $f$  is of *isotype*  $\tau$  if  $f$  is an element of an irreducible  $S_N$ -submodule of  $\mathcal{P}_n$  on which the representation  $\tau$  is realized. It was conjectured in [7] that the two-part representations  $(\mu, N - \mu)$  (with  $2\mu \geq N$ ) give rise to singular polynomials for the singular values  $-\frac{m}{\mu+1}$  with  $\gcd(m, \mu+1) < \frac{\mu+1}{N-\mu}$  (this was shown in [5]), and the representations  $(s(\mu+1) - 1, \mu, \dots, \mu, \rho)$  for  $s, \mu \in \mathbb{N}$  give rise to singular polynomials for the singular values  $-\frac{m}{\mu+1}$  with  $\gcd(m, \mu+1) = 1$ . The latter is the main topic of this paper. For example, the singular values  $-\frac{m}{6}$  for  $N = 10$  are associated with the isotypes  $(5, 5)$  for  $m \equiv 1, 5 \pmod{6}$ ,  $(5, 2, 2, 1)$  for  $m \equiv 2, 4 \pmod{6}$ , and  $(5, 1^5)$  for  $m \equiv 3 \pmod{6}$ .

In the rest of this introduction we present definitions and key properties of nonsymmetric Jack polynomials, hook-length products for Ferrers diagrams, and the fundamental partial order on compositions. Section 2 contains detailed formulae for the action of  $\{\mathcal{D}_i\}$  on the polynomials, with emphasis on the poles. The construction of singular polynomials is presented in Section 3, and there is a key result on the absence of certain poles when the number of variables (that is,  $N$ ) is small enough. Murphy's construction [13] of the seminormal representations of  $S_N$  is used in Section 4 to analyze the irreducible  $S_N$ -modules generated by singular polynomials. The conclusion in Section 5 concisely displays the correspondence between pairs  $(m, n)$ ,  $2 \leq n \leq N$ ,  $\frac{m}{n} \notin \mathbb{Z}$  and singular polynomials for  $\kappa = -\frac{m}{n}$ , and also considers modules of the specializations of the rational Cherednik algebra, defined in terms of singular polynomials.

Our construction will be in terms of nonsymmetric Jack polynomials. Since these have coefficients in  $\mathbb{Q}(\kappa)$  with poles at negative rational values of  $\kappa$  it will be important to be precise about these poles. Any further reference to poles will be with respect to  $\kappa$ . The related commutative algebra of self-adjoint operators is

generated by

$$\mathcal{U}_i f(x) = \mathcal{D}_i x_i f(x) - \kappa \sum_{j=1}^{i-1} (j, i) f(x), 1 \leq i \leq N.$$

(this differs by an additive constant from the notation in [8, Ch.8]). The operators act in a triangular manner on monomials.

**Definition 2.** For  $\alpha \in \mathbb{N}_0^N$  let  $\alpha^+$  denote the unique partition such that  $\alpha^+ = w\alpha$  for some  $w \in S_N$ . For  $\alpha, \beta \in \mathbb{N}_0^N$  the partial order  $\alpha \succ \beta$  ( $\alpha$  dominates  $\beta$ ) means that  $\alpha \neq \beta$  and  $\sum_{i=1}^j \alpha_i \geq \sum_{i=1}^j \beta_i$  for  $1 \leq j \leq N$ ; and  $\alpha \triangleright \beta$  means that  $|\alpha| = |\beta|$  and either  $\alpha^+ \succ \beta^+$  or  $\alpha^+ = \beta^+$  and  $\alpha \succ \beta$ . The notations  $\alpha \succeq \beta$  and  $\alpha \geq \beta$  include the case that  $\alpha = \beta$ .

Acting on the monomial basis of  $\mathcal{P}_n$  the operators  $\mathcal{U}_i$  have on-diagonal coefficients involving the following “rank” function on  $\mathbb{N}_0^N$ . We denote the cardinality of a set  $E$  by  $\#E$ .

**Definition 3.** For  $\alpha \in \mathbb{N}_0^N$  and  $1 \leq i \leq N$  let

$$\begin{aligned} r(\alpha, i) &= \#\{j : \alpha_j > \alpha_i\} + \#\{j : 1 \leq j \leq i, \alpha_j = \alpha_i\}, \\ \xi_i(\alpha) &= (N - r(\alpha, i))\kappa + \alpha_i + 1. \end{aligned}$$

Clearly for a fixed  $\alpha \in \mathbb{N}_0^N$  the values  $\{r(\alpha, i) : 1 \leq i \leq N\}$  consist of all of  $\{1, \dots, N\}$ , are independent of trailing zeros (that is, if  $\alpha' \in \mathbb{N}_0^M$ ,  $\alpha'_i = \alpha_i$  for  $1 \leq i \leq N$  and  $\alpha'_i = 0$  for  $N < i \leq M$  then  $r(\alpha, i) = r(\alpha', i)$  for  $1 \leq i \leq N$ ), and  $\alpha \in \mathbb{N}_0^{N,P}$  if and only if  $r(\alpha, i) = i$  for all  $i$ . Then (see [8, p.291])  $\mathcal{U}_i x^\alpha = \xi_i(\alpha) x^\alpha + q_{\alpha, i}(x)$  where  $q_{\alpha, i}(x)$  is a sum of terms  $\pm \kappa x^\beta$  with  $\alpha \triangleright \beta$ . The nonsymmetric Jack polynomials are the simultaneous eigenvectors of  $\{\mathcal{U}_i : 1 \leq i \leq N\}$ , well-defined for generic  $\kappa$ . Opdam [14, p.83] discovered and studied them in the wider framework of polynomials associated to crystallographic root systems. There are two useful normalizations of these polynomials, one is “monic in  $x$ ” the other is “monic in  $p$ ”. The  $p$ -basis is defined by the generating function

$$\sum_{\alpha \in \mathbb{N}_0^N} p_\alpha(x) y^\alpha = \prod_{i=1}^N \left( (1 - x_i y_i)^{-1} \prod_{j=1}^N (1 - x_i y_j)^{-\kappa} \right), \text{ for } \max_{i,j} |x_i| |y_j| < 1.$$

In contrast to the monomial basis  $\mathcal{U}_i p_\alpha = \xi_i(\alpha) p_\alpha + q'_{\alpha, i}$  where  $q'_{\alpha, i}$  is a sum of terms  $\pm \kappa p_\beta$  with  $\beta \triangleright \alpha$  (and  $\ell(\beta) \leq \ell(\alpha)$ ), (see [8, Prop. 8.4.11]).

**Definition 4.** For  $\alpha \in \mathbb{N}_0^N$  let  $\zeta_\alpha, \zeta_\alpha^x$  denote the  $p$ -monic and  $x$ -monic, respectively, simultaneous eigenvectors, that is,  $\mathcal{U}_i \zeta_\alpha = \xi_i(\alpha) \zeta_\alpha$ ,  $\mathcal{U}_i \zeta_\alpha^x = \xi_i(\alpha) \zeta_\alpha^x$  for  $1 \leq i \leq N$  and  $\zeta_\alpha = p_\alpha + \sum_{\beta \triangleright \alpha} A_{\beta\alpha} p_\beta$ ,  $\zeta_\alpha^x = x^\alpha + \sum_{\alpha \triangleright \beta} A_{\beta\alpha}^x x^\beta$ , with coefficients  $A_{\beta\alpha}, A_{\beta\alpha}^x \in \mathbb{Q}(\kappa)$ .

Suppose that  $\ell(\alpha) = m$  for some  $m \geq 1$  then the coefficients  $A_{\beta\alpha}$  do not depend on  $N \geq m$  (and  $A_{\beta\alpha} \neq 0$  implies  $\ell(\beta) \leq m$ ); on the other hand, if  $\beta \in \mathbb{N}_0^N$  and  $\ell(\beta) \leq m$  then  $A_{\beta\alpha}^x$  does not depend on  $N \geq m$  (that is, if  $N > M \geq m$  then the projection of  $\mathbb{R}^N$  onto  $\mathbb{R}^M$  setting  $x_{M+1} = \dots = x_N = 0$  and annihilating the terms  $A_{\beta\alpha}^x x^\beta$  with  $\beta_i \neq 0$  for some  $i > M$  ( $\ell(\beta) > m$ ), produces the  $\mathbb{R}^M$ -polynomial. The relation between the two types involves hook-length products. Suppose  $\lambda \in \mathbb{N}_0^{N,P}$  and  $\ell(\lambda) = m$ ; the Ferrers diagram of  $\lambda$  is the set  $\{(i, j) : 1 \leq i \leq m, 1 \leq j \leq \lambda_i\}$ .

Each node  $(i, j)$  has the *arm*  $\{(i, l) : j < l \leq \lambda_i\}$  and the *leg*  $\{(l, j) : i < l, j \leq \lambda_l\}$ . The node itself, the arm and the leg make up the *hook*. For  $t \in \mathbb{Q}(\kappa)$  the *hook-length*, the hook-length product and generalized Pochhammer symbol for  $\lambda$  are given by

$$h(\lambda, t; i, j) = (\lambda_i - j + t + \kappa \# \{l : i < j, j \leq \lambda_l\})$$

$$h(\lambda, t) = \prod_{i=1}^m \prod_{j=1}^{\lambda_i} h(\lambda, t; i, j),$$

$$(t)_\lambda = \prod_{i=1}^m \prod_{j=1}^{\lambda_i} (t - (i-1)\kappa + j - 1).$$

The coordinate-wise notation for hook-lengths will appear in the context of specializations of  $\kappa$  to negative rational numbers. The compositions  $\alpha \in \mathbb{N}_0^N$  are associated with the products

$$\mathcal{E}_\varepsilon(\alpha) = \prod \left\{ 1 + \frac{\varepsilon \kappa}{\kappa(r(\alpha, i) - r(\alpha, j)) + \alpha_j - \alpha_i} : i < j, \alpha_i < \alpha_j \right\}, \varepsilon = \pm.$$

Note that the denominator is identical to  $\xi_j(\alpha) - \xi_i(\alpha)$  and  $\mathcal{E}_\varepsilon(\lambda) = 1$  for  $\lambda \in \mathbb{N}_0^{N,P}$ . Then (see [8, p.323]) for each  $\alpha \in \mathbb{N}_0^N$

$$\zeta_\alpha = \mathcal{E}_+(\alpha) \mathcal{E}_-(\alpha) \frac{h(\alpha^+, \kappa + 1)}{h(\alpha^+, 1)} \zeta_\alpha^x.$$

Also  $\zeta_\alpha(1^N) = \mathcal{E}_-(\alpha) (N\kappa + 1)_{\alpha^+} / h(\alpha^+, 1)$ . Knop and Sahi [11] by finding explicit combinatorial formulae in terms of tableaux established the key theorem that  $h(\lambda, \kappa + 1) \zeta_\lambda^x$  is a polynomial with coefficients in  $\mathbb{Z}[\kappa]$ . However as  $N$  decreases the set of  $\kappa$ -poles of  $\zeta_\lambda^x$  also decreases, and specific results will be established and used in the sequel (by [11, Cor. 4.7] the coefficient of  $x_{m+1} \dots x_{m+n}$  in  $\zeta_\lambda^x$  is  $n! \kappa^n / h(\lambda, \kappa + 1)$  where  $|\lambda| = n$  and  $\ell(\lambda) = m$ ; thus for  $m \leq N < m + n$  one expects some poles to be omitted). An obvious sufficient condition for the presence of a pole in  $\zeta_\lambda^x$  for given  $N$  is its presence in  $\zeta_\lambda^x(1^N) = (N\kappa + 1)_\lambda / h(\lambda, \kappa + 1)$ .

We can now state our main results: for each isotype  $\tau$  and singular value  $\kappa_0$  the corresponding singular polynomials form the  $S_N$ -module generated by  $\zeta_\lambda^x$  for a certain  $\lambda$ , that is,  $\text{span}_{\mathbb{Q}} \{w \zeta_\lambda^x : w \in S_N\}$ ; (in fact a basis will be specified in terms of reverse lattice permutations of  $\lambda$ )

- for  $\tau = (\mu, N - \mu), \kappa_0 = -\frac{m}{\mu+1}$  with  $\gcd(m, \mu + 1) < \frac{\mu+1}{N-\mu}$ , let  $\lambda = (m^{N-\mu}, 0^\mu)$  (that is,  $m$  is repeated  $N - \mu$  times followed by  $\mu$  zeros)
- for  $\tau = (s(\mu + 1) + \mu, \mu^l, \rho)$  where  $l \geq 1, s \geq 0$ , and  $1 \leq \rho \leq \mu$  (so that  $N = (s + l + 1)\mu + s + \rho$ ),  $\kappa_0 = -\frac{m}{\mu+1}$  with  $\gcd(m, \mu + 1) = 1$ , let  $\lambda = ((m(s + l + 1))^\rho, (m(s + l))^\mu, \dots, (m(s + 1))^\mu, 0^{s(\mu+1)+\mu})$ .

For example, let  $N = 10, \tau = (5, 2, 2, 1), \kappa_0 = -\frac{m}{3}$  and  $\gcd(m, 3) = 1$ , then  $\lambda = (4m, 3m, 3m, 2m, 2m, 0^5)$ . The singular polynomials for  $N = 2k + 1, \tau = (2k - 1, 1, 1), \kappa_0 = -\frac{m}{2}, \lambda = (m(k + 1), mk)$  were found in [6] by a different method (not suitable for the general problem).

## 2. DIFFERENTIATION FORMULAE

This section contains expressions for  $\mathcal{D}_i \zeta_\alpha$  in terms of  $\{\zeta_\beta : |\beta| = |\alpha| - 1\}$ , valid for generic  $\kappa$ . There is some material dealing with  $x$ -monic polynomials, however

the  $p$ -monic polynomials have somewhat simpler formulae. The basic step is the formula for  $\mathcal{D}_m \zeta_\alpha$  where  $\ell(\alpha) = m$ ,  $\alpha \in \mathbb{N}_0^N$ ; further from properties of the  $p$ -basis it follows that  $\mathcal{D}_i \zeta_\alpha = 0$  for  $i > m$ . The computation involves a cyclic shift. For  $1 \leq i \leq j \leq N$  let  $[i, j]$  denote the interval  $\{k \in \mathbb{Z} : i \leq k \leq j\}$  and let  $S[i, j]$  denote the subgroup of  $S_N$  generated by  $\{(i, i+1), (i+1, i+2), \dots, (j-1, j)\}$  (isomorphic to  $S_{j+1-i}$ ).

**Definition 5.** For  $1 < m \leq N$  let  $\theta_m = (1, 2)(2, 3) \dots (m-1, m) \in S_N$ , and if  $\alpha \in \mathbb{N}_0^N$  satisfies  $\ell(\alpha) = m$  then  $\tilde{\alpha} = \theta_m(\alpha - \varepsilon(m)) = (\alpha_m - 1, \alpha_1, \dots, \alpha_{m-1}, 0, \dots)$ .

**Lemma 1.** Suppose  $\alpha \in \mathbb{N}_0^N$  satisfies  $\ell(\alpha) = m$  then (i)  $\mathcal{U}_1 \theta_m \mathcal{D}_m \zeta_\alpha = (\xi_m(\alpha) - 1) \theta_m \mathcal{D}_m \zeta_\alpha$ , (ii)  $\mathcal{U}_i \theta_m \mathcal{D}_m \zeta_\alpha = \xi_{i-1}(\alpha) \theta_m \mathcal{D}_m \zeta_\alpha$  for  $1 < i \leq m$ , and (iii)  $\mathcal{U}_i \theta_m \mathcal{D}_m \zeta_\alpha = ((N-i)\kappa + 1) \theta_m \mathcal{D}_m \zeta_\alpha$  for  $i > m$ .

*Proof.* The commutation  $(x_m \mathcal{D}_m - \mathcal{D}_m x_m) f = -f - \kappa \sum_{j \neq m} (j, m) f$  ([8, p.290]) shows that

$$\begin{aligned} \mathcal{D}_m x_m \mathcal{D}_m \zeta_\alpha &= -\mathcal{D}_m \zeta_\alpha + \mathcal{D}_m \left( \mathcal{D}_m x_m - \kappa \sum_{j < m} (j, m) \right) \zeta_\alpha - \kappa \sum_{j > m} \mathcal{D}_m (j, m) \zeta_\alpha \\ &= \mathcal{D}_m (\xi_m(\alpha) - 1) \zeta_\alpha \end{aligned}$$

because  $\mathcal{D}_m (j, m) \zeta_\alpha = (j, m) \mathcal{D}_j \zeta_\alpha = 0$  for  $j > m$ . Apply  $\theta_m$  to the previous equation to prove part (i) (since  $\theta_m \mathcal{D}_m x_m = \mathcal{D}_1 x_1 \theta_m$ ). Next suppose that  $1 < i \leq m$  then  $\theta_m^{-1} \mathcal{U}_i \theta_m = \mathcal{D}_{i-1} x_{i-1} - \kappa \sum_{j=1}^{i-2} (j, i-1) - \kappa (m, i-1)$ . Apply this operator to  $\mathcal{D}_m \zeta_\alpha$  to obtain

$$\begin{aligned} \theta_m^{-1} \mathcal{U}_i \theta_m \mathcal{D}_m \zeta_\alpha &= \mathcal{D}_m \left( \mathcal{D}_{i-1} x_{i-1} - \kappa \sum_{j=1}^{i-2} (j, i-1) \right) \zeta_\alpha \\ &\quad + \kappa (\mathcal{D}_{i-1} (i-1, m) - (m, i-1) \mathcal{D}_m) \\ &= \mathcal{D}_m \xi_{i-1}(\alpha) \zeta_\alpha. \end{aligned}$$

The computation uses the commutativity of  $\mathcal{D}_m$  and  $\mathcal{D}_{i-1}$  and the commutation  $(x_j \mathcal{D}_m - \mathcal{D}_m x_j) f = \kappa (j, m) f$  ([8, p.290]) for  $j \neq m$ . This shows part (ii). Similarly for  $i > m$  we have that  $\theta_m^{-1} \mathcal{U}_i \theta_m = \mathcal{U}_i$  and  $\mathcal{U}_i \mathcal{D}_m - \mathcal{D}_m \mathcal{U}_i = \kappa (\mathcal{D}_i (i, m) - (i, m) \mathcal{D}_m + \mathcal{D}_m (i, m)) = \kappa (i, m) \mathcal{D}_i$ . But  $\mathcal{D}_i \zeta_\alpha = 0$  for  $i > m$  and so  $\mathcal{U}_i \mathcal{D}_m \zeta_\alpha = \xi_i(\alpha) \mathcal{D}_m \zeta_\alpha$  and  $\xi_i(\alpha) = (N-i)\kappa + 1$ ; proving part (iii).  $\square$

The following is used to pick out a coefficient in  $\mathcal{D}_m \zeta_\alpha$ .

**Lemma 2.** Suppose  $\alpha, \beta \in \mathbb{N}_0^N$ ,  $|\alpha| = |\beta|$  and  $\ell(\alpha) = \ell(\beta) = m$ , if  $p_{\alpha - \varepsilon(m)}$  appears with a nonzero coefficient in the expansion of  $\mathcal{D}_m p_\beta$  then either  $\alpha = \beta$  or  $\beta \triangleleft \alpha$ . If  $\alpha = \beta$  then the coefficient is  $(N+1-r(\alpha, m))\kappa + \alpha_m$ .

*Proof.* By Prop.8.4.3 [8, p.294]

$$\begin{aligned} \mathcal{D}_m p_\beta &= ((N - \#\{j : \beta_j \geq \beta_m\} + 1)\kappa + \beta_m) p_{\beta - \varepsilon(m)} \\ &\quad + \kappa \sum \{p_\gamma : \gamma = \beta + n\varepsilon(m) - (n+1)\varepsilon(j), \max(0, \beta_j - \beta_m) \leq n \leq \beta_j - 1, j \neq m\} \\ &\quad - \kappa \sum \{p_\gamma : \gamma = \beta - (n+1)\varepsilon(m) + n\varepsilon(j), \max(1, \beta_m - \beta_j) \leq n \leq \beta_m - 1, j \neq m\}. \end{aligned}$$

If  $\beta = \alpha$  then the coefficient of  $p_{\alpha - \varepsilon(m)}$  is  $(N - r(\alpha, m) + 1)\kappa + \alpha_m$ ; note that  $j > m$  implies  $\alpha_j = 0 < \alpha_m$ . If  $p_{\alpha - \varepsilon(m)}$  has the coefficient  $\kappa$  then  $\beta = \alpha -$

$(n+1)(\varepsilon(m) - \varepsilon(j))$  and the restrictions are equivalent to  $0 \leq n \leq \alpha_m - \alpha_j - 2$  (for some  $j \neq m$ ); thus  $n+1 < \alpha_m - \alpha_j$  and  $\beta^+ \prec \alpha^+$  by Lemma 8.2.3(iv) [8, p.289]. If  $p_{\alpha-\varepsilon(m)}$  has the coefficient  $-\kappa$  then  $\beta = \alpha - n(\varepsilon(j) - \varepsilon(m))$  and the restrictions are equivalent to  $1 \leq n \leq \alpha_j - \alpha_m$  (for some  $j \neq m$ ). If  $n < \alpha_j - \alpha_m$  then by the same lemma  $\beta^+ \prec \alpha^+$ . If  $n = \alpha_j - \alpha_m$  then  $\beta = (j, m)\alpha$  and  $\beta \prec \alpha$  because  $\alpha_j > \alpha_m$  and  $j < m$  (using the hypothesis  $\ell(\alpha) = m$ ). Thus, if  $\alpha \neq \beta$  then  $\beta \triangleleft \alpha$ .  $\square$

**Theorem 1.** Suppose  $\alpha \in \mathbb{N}_0^N$  and  $\ell(\alpha) = m$ , then

$$\mathcal{D}_m \zeta_\alpha = ((N+1-r(\alpha, m))\kappa + \alpha_m) \theta_m^{-1} \zeta_{\tilde{\alpha}}.$$

*Proof.* By Lemma 1,  $\theta_m \mathcal{D}_m \zeta_\alpha$  is a simultaneous eigenvector of  $\{\mathcal{U}_i : 1 \leq i \leq N\}$  with eigenvalues  $(\xi_m(\alpha) - 1, \xi_1(\alpha), \dots, \xi_{m-1}(\alpha), \xi_{m+1}(\alpha), \dots)$ . We claim these are the eigenvalues for  $\tilde{\alpha}$ . Indeed  $r(\tilde{\alpha}, 1) = \#\{j : \alpha_j > \alpha_m - 1, j < m\} + 1 = \#\{j : \alpha_j \geq \alpha_m\} = r(\alpha, m)$ . For any values  $\alpha_i$  different from  $\alpha_m - 1$  and 0 it is obvious that  $r(\alpha, i) = r(\tilde{\alpha}, i+1)$ . Suppose for some  $i < m$  that  $\alpha_i = \alpha_m - 1$  then

$$\begin{aligned} r(\tilde{\alpha}, i+1) &= \#\{j : \alpha_j > \alpha_m - 1, j < m\} + \#\{j : j \leq i, \alpha_j = \alpha_m - 1\} + 1 \\ &= \#\{j : \alpha_j > \alpha_m - 1, j \leq m\} + \#\{j : j \leq i, \alpha_j = \alpha_m - 1\} \\ &= r(\alpha, i). \end{aligned}$$

For  $i > m$ , obviously  $\xi_i(\alpha) = \xi_i(\tilde{\alpha}) = (N-i)\kappa + 1$ . Thus  $\theta_m \mathcal{D}_m \zeta_\alpha = c \zeta_{\tilde{\alpha}}$  for some constant  $c$ , which will be determined by finding the coefficient of  $\theta_m^{-1} p_{\tilde{\alpha}} = p_{\alpha-\varepsilon(m)}$  in  $\mathcal{D}_m \zeta_\alpha$ . Since  $\zeta_\alpha = p_\alpha + \sum_{\beta \triangleright \alpha} A_{\beta\alpha} p_\beta$  (and  $\ell(\beta) \leq m$ ) we obtain  $\mathcal{D}_m \zeta_\alpha = \mathcal{D}_m p_\alpha + \sum_{\beta \triangleright \alpha} A_{\beta\alpha} \mathcal{D}_m p_\beta$ . If  $p_{\alpha-\varepsilon(m)}$  has a nonzero coefficient in  $\mathcal{D}_m p_\beta$  then  $\ell(\beta) = m$  (else  $\beta_m = 0$  and  $\mathcal{D}_m p_\beta = 0$ ) and by Lemma 2  $\beta = \alpha$  or  $\beta \triangleright \alpha$ . only the case  $\beta = \alpha$  can occur and thus  $c = (N+1-r(\alpha, m))\kappa + \alpha_m$ .  $\square$

With the intention of using the theorem to compute arbitrary  $\mathcal{D}_i \zeta_\lambda$  with  $\lambda \in \mathbb{N}_0^{N,P}$  we observe that it suffices to consider the points of decrease, that is,  $\lambda_i > \lambda_{i+1}$ , (the values of  $i$  for which  $\lambda - \varepsilon(i)$  is a partition) then apply the transpositions  $(j, j+1)$  successively for  $j = i, i+1, \dots, \ell(\lambda) - 1$ , apply  $\mathcal{D}_m$ , with a result involving  $\zeta_\alpha$  where  $\alpha = (\lambda_i - 1, \lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots)$  (this is an over-simplification; actually all the points of decrease between  $i$  and  $\ell(\lambda)$  must be considered). Finally transform  $\zeta_{\lambda-\varepsilon(i)}$  to  $\zeta_\alpha$  with another sequence of transpositions. As mentioned before it is necessary to keep track of the  $\kappa$ -poles occurring in these operations. The basic step is the action of an adjacent transposition on  $\zeta_\alpha$ .

**Proposition 1.** Suppose  $\alpha \in \mathbb{N}_0^N$ , and  $\alpha_i > \alpha_{i+1}$  for some  $i$ , then let  $\sigma = (i, i+1)$  and  $a = \kappa((r(\alpha, i+1) - r(\alpha, i))\kappa + \alpha_i - \alpha_{i+1})^{-1}$  then  $\zeta_{\sigma\alpha} = \sigma\zeta_\alpha - a\zeta_\alpha$  and  $\zeta_{\sigma\alpha}^x = \frac{1}{1-a^2}(\sigma\zeta_\alpha^x - a\zeta_\alpha^x)$ .

The proof for the  $p$ -monic case is in Prop. 8.5.5 [8, p.301]; the proof for the  $x$ -monic case can be deduced from the inverse of the  $p$ -monic formula and the equation  $\zeta_\alpha^x = \sigma\zeta_{\sigma\alpha}^x + a\zeta_{\sigma\alpha}^x$  (arguing that  $x^\alpha$  does not appear in  $\zeta_{\sigma\alpha}^x$  since  $\alpha \triangleright \sigma\alpha$ ). Note that the denominator  $(r(\alpha, i+1) - r(\alpha, i))\kappa + \alpha_i - \alpha_{i+1} = \xi_i(\alpha) - \xi_{i+1}(\alpha)$ . For singular values of  $\kappa$  it can happen that  $a = -1$  implying that  $\sigma\zeta_\alpha^x = -\zeta_\alpha^x$ . We need an extension of the proposition applying to the situation of several adjacent entries of  $\alpha$  being equal.

**Proposition 2.** Suppose  $\alpha \in \mathbb{N}_0^N$  with  $\alpha_i = a > b = \alpha_{i+j}$  for  $1 \leq j \leq s$ , where  $1 \leq i < i + s \leq N$ , then

$$\zeta_{(i,i+s)\alpha} = (i, i + s) \zeta_\alpha - \frac{\kappa}{(r(\alpha, i + s) - r(\alpha, i)) \kappa + a - b} \left( 1 + \sum_{j=1}^{s-1} (i, i + j) \right) \zeta_\alpha.$$

*Proof.* Observe that  $(i, i + s) \alpha = (\dots, b, \dots, b, a, \dots)$ . The proof is by induction on  $s$  and depends on the invariance of  $\zeta_\alpha$  under the subgroup  $S[i + 1, i + s]$ . Since  $r(\alpha, i + j) = r(\alpha, i + 1) + j - 1$  for  $1 \leq j \leq s$  let  $C = (r(\alpha, i + 1) - r(\alpha, i) - 1) \kappa + a - b$  so that  $(r(\alpha, i + j) - r(\alpha, i)) \kappa + a - b = j\kappa + C$ , also let  $f_j = \zeta_{(i,i+j)\alpha}$  and  $c_j = -\frac{\kappa}{j\kappa + C}$ . By Proposition 1  $f_{j+1} = (i + j, i + j + 1) f_j + c_{j+1} f_j$ . By the inductive hypothesis  $f_j = (i, i + j) \zeta_\alpha + c_j \left( 1 + \sum_{k=1}^{j-1} (i, i + k) \right) \zeta_\alpha$ . Thus

$$\begin{aligned} f_{j+1} &= (i + j, i + j + 1) (i, i + j) \zeta_\alpha + c_{j+1} (i, i + j) \zeta_\alpha \\ &\quad + c_j ((i + j, i + j + 1) + c_{j+1}) \left( 1 + \sum_{k=1}^{j-1} (i, i + k) \right) \zeta_\alpha \\ &= (i, i + j) \zeta_\alpha + c_{j+1} (i, i + j) \zeta_\alpha \\ &\quad + c_j \left( 1 + \sum_{k=1}^{j-1} (i, i + k) \right) ((i + j, i + j + 1) + c_{j+1}) \zeta_\alpha \\ &= (i, i + j) \zeta_\alpha + c_{j+1} (i, i + j) \zeta_\alpha + c_j (1 + c_{j+1}) \left( 1 + \sum_{k=1}^{j-1} (i, i + k) \right) \zeta_\alpha. \end{aligned}$$

By the invariance property of  $\zeta_\alpha$  we have  $(i + j, i + j + 1) (i, i + j) \zeta_\alpha = (i + j, i + j + 1) (i, i + j) (i + j, i + j + 1) \zeta_\alpha = (i, i + j) \zeta_\alpha$ . Furthermore  $c_j (1 + c_{j+1}) = -\frac{\kappa}{j\kappa + C} \left( 1 - \frac{\kappa}{j\kappa + \kappa + C} \right) = c_{j+1}$  and this completes the induction.  $\square$

There is a similar result for the opposite direction.

**Proposition 3.** Suppose  $\alpha \in \mathbb{N}_0^N$  with  $\alpha_{i+j} = b > a = \alpha_{i+s}$  for  $0 \leq j \leq s - 1$ , where  $1 \leq i < i + s \leq N$ , then

$$\zeta_{(i,i+s)\alpha} = (i, i + s) \zeta_\alpha - \frac{\kappa}{(r(\alpha, i + s) - r(\alpha, i)) \kappa + b - a} \left( 1 + \sum_{j=1}^{s-1} (i + j, i + s) \right) \zeta_\alpha.$$

*Proof.* Proceeding similarly to the previous case, for  $1 \leq j \leq s$ ,  $r(\alpha, i + s - j) = r(\alpha, i + s - 1) + 1 - j$  and let  $f_j = \zeta_{(i,i+s-j)\alpha}$  and  $c_j = -\kappa ((r(\alpha, i + s) - r(\alpha, i + s - 1) - 1 + j) \kappa + b - a)^{-1}$ . Also  $\zeta_\alpha$  is invariant under  $S[i, i + s - 1]$ . The inductive step is based on  $f_{j+1} = (i + s - j - 1, i + s - j) f_j + c_{j+1} f_j$  (and  $f_0 = \zeta_\alpha$ ). The rest of the argument is similar to the previous one and is omitted.  $\square$

To illustrate the basic step, apply  $\mathcal{D}_{i+s}$  to both sides of the formula in Proposition 2 and obtain

$$\begin{aligned} \mathcal{D}_{i+s}\zeta_{(i,i+s)\alpha} &= (i, i+s) \mathcal{D}_i \zeta_\alpha \\ &\quad - \frac{\kappa}{(r(\alpha, i+s) - r(\alpha, i)) \kappa + a - b} \left( 1 + \sum_{j=1}^{s-1} (i, i+j) \right) \mathcal{D}_{i+s} \zeta_\alpha, \\ \mathcal{D}_i \zeta_\alpha &= (i, i+s) \mathcal{D}_{i+s} \zeta_{(i,i+s)\alpha} \\ &\quad + \frac{\kappa}{(r(\alpha, i+s) - r(\alpha, i)) \kappa + a - b} (i, i+s) \left( 1 + \sum_{j=1}^{s-1} (i, i+j) \right) \mathcal{D}_{i+s} \zeta_\alpha; \end{aligned}$$

so that the index for  $\mathcal{D}$  is increased (eventually to  $m$ ).

Fix a partition  $\lambda \in \mathbb{N}_0^{N,P}$ , suppose that the parts of  $\lambda$  have  $M$  distinct nonzero values, the points of decrease are  $i_1 < i_2 < \dots < i_M$ , so that  $\lambda_i$  is constant on each interval  $i_{j-1} < i \leq i_j$  (interpret  $i_0 = 0$ , also let  $\ell(\lambda) = m = i_M$ ). For  $1 \leq j < k \leq M$  let

$$\begin{aligned} C_{jk} &= \frac{\kappa}{(i_k - i_j) \kappa + \lambda_{i_j} - \lambda_{i_k}}, \\ w_j &= 1 + \sum_{r=i_j+1}^{i_{j+1}-1} (i_j, r) \in \mathbb{Z}S[i_j, i_{j+1}-1], \\ z_{jk} &= (i_{k-1}, i_k) - C_{jk} w_{k-1}, \end{aligned}$$

further let  $\mu(j, k) \in \mathbb{N}_0^N$  be the action on  $\lambda$  by the cyclic shift on the interval  $\{i_j, \dots, i_k\}$ , that is  $\mu(j, k)_{i_k} = \lambda_{i_j}$ ,  $\mu(j, k)_i = \lambda_{i+1}$  for  $i_j \leq i < i_k$  and  $\mu(j, k)_i = \lambda_i$  for  $i < i_j$  or  $i > i_k$ . Proposition 2 applies to the transformation of  $\zeta_{\mu(j, k)}$  to  $\zeta_{\mu(j, k+1)}$ ; note that  $r(\mu(j, k), i_k) = i_j$  and  $r(\mu(j, k), i_{k+1}) = i_{k+1}$  thus  $\zeta_{\mu(j, k+1)} = z_{j, k+1} \zeta_{\mu(j, k)}$ . The start of this recurrence is  $\zeta_{\mu(j, j)} = \zeta_\lambda$ . The object is to express any  $\mathcal{D}_i \zeta_\lambda$  in terms of  $\mathcal{D}_m \zeta_{\mu(j, M)}$ ,  $j = 1, \dots, M$ . It suffices to consider  $\{\mathcal{D}_{i_j} \zeta_\lambda\}$  since  $\mathcal{D}_i \zeta_\lambda = (i, i_j) \mathcal{D}_{i_j} \zeta_\lambda$  for  $i_{j-1} < i < i_j$ .

**Lemma 3.** For  $k = 1, \dots, M - j$

$$\begin{aligned} \mathcal{D}_{i_M} z_{j, M} \dots z_{j, M-k+1} &= (i_{M-1}, i_M) (i_{M-2}, i_{M-1}) \dots (i_{M-k}, i_{M-k+1}) \mathcal{D}_{i_{M-k}} \\ &\quad - \sum_{s=0}^{k-1} C_{j, M-s} (i_{M-1}, i_M) \dots (i_{M-s}, i_{M-s+1}) w_{M-s-1} z_{j, M-s-1} \dots z_{j, M-k+1} \mathcal{D}_{i_{M-s}}. \end{aligned}$$

*Proof.* We proceed by induction. The formula is tautological for  $k = 0$ . Also the term in the sum with  $s = k - 1$  has no  $z_{jn}$  factors. Multiply the right hand side by  $z_{j, M-k}$  on the right. For the first part,  $\mathcal{D}_{i_{M-k}} ((i_{M-k-1}, i_{M-k}) - C_{j, M-k} w_{M-k-1}) = (i_{M-k-1}, i_{M-k}) \mathcal{D}_{i_{M-k-1}} - C_{j, M-k} w_{M-k-1} \mathcal{D}_{i_{M-k}}$  (since  $\mathcal{D}_{i_n}$  commutes with  $w_j$  for  $n \neq j$ ). For the second part,  $\mathcal{D}_{i_{M-s}}$  commutes with  $z_{j, M-k}$ . This completes the induction.  $\square$

Set  $k = M - j$  in the lemma, apply the operator to  $\zeta_\lambda$  and multiply both sides of the identity by  $(i_j, i_{j+1}) \dots (i_{M-1}, i_M)$ , yielding (replace  $s$  by  $M - s$ )

$$\begin{aligned} \mathcal{D}_{i_j} \zeta_\lambda &= (i_j, i_{j+1}) \dots (i_{M-1}, i_M) \mathcal{D}_m \zeta_{\mu(j, M)} \\ &+ \sum_{s=j+1}^M C_{j, s} (i_j, i_{j+1}) \dots (i_{s-1}, i_s) w_{s-1} z_{j, s-1} \dots z_{j, j+1} \mathcal{D}_{i_s} \zeta_\lambda. \end{aligned}$$

This identity is used starting with  $j = M - 1$  and then decrementing  $j$  by 1 with the result:

$$\mathcal{D}_{i_j} \zeta_\lambda = (i_j, i_{j+1}) \dots (i_{M-1}, i_M) \mathcal{D}_m \zeta_{\mu(j, M)} + \sum_{s=j+1}^M u_{j, s} \mathcal{D}_m \zeta_{\mu(s, M)},$$

where each  $u_{j, s} \in RS[i_1, m]$  and  $R$  is the  $\mathbb{Z}$ -ring generated by  $\{C_{j, k} : 1 \leq j < k \leq M\}$ . To complete the analysis of  $\mathcal{D}_m \zeta_{\mu(j, M)}$ , for  $0 \leq k < j$  let  $\nu(k, j) \in \mathbb{N}_0^N$  be the action on  $\lambda - \varepsilon(i_j)$  by the (reverse) cyclic shift on the interval  $\{i_k + 1, \dots, i_j\}$ , that is  $\nu(k, j)_{i_{k+1}} = \lambda_{i_j} - 1$ ,  $\mu(j, k)_i = \lambda_{i-1}$  for  $i_k + 1 < i \leq i_j$  and  $\nu(k, j)_i = \lambda_i$  for  $i \leq i_k$  or  $i > i_j$ . Also let  $\nu(j, j) = \lambda - \varepsilon(i_j)$ ; if  $i_j = i_{j-1} + 1$  then  $\nu(j-1, j) = \nu(j, j)$ . For  $0 \leq k < j-1 < M$  let

$$\begin{aligned} C'_{kj} &= \frac{\kappa}{(i_j - i_k - 1) \kappa + \lambda_{i_{k+1}} - \lambda_{i_j} + 1}, \\ w'_k &= 1 + \sum_{r=i_k+2}^{i_{k+1}} (r, i_{k+1} + 1) \in \mathbb{Z}S[i_k + 2, i_{k+1} + 1], \end{aligned}$$

and if  $i_{j-1} < i_j - 1$  let

$$\begin{aligned} C'_{j-1, j} &= \frac{\kappa}{(i_j - i_{j-1} - 1) \kappa + 1}, \\ w'_{j-1} &= 1 + \sum_{r=i_{j-1}+2}^{i_j-1} (r, i_j) \in \mathbb{Z}S[i_{j-1} + 2, i_j]. \end{aligned}$$

Proposition 3 applies to the transformation of  $\zeta_{\nu(k, j)}$  to  $\zeta_{\nu(k-1, j)}$ ; note that  $r(\nu(k, j), i_k + 1) = i_k + 1$  and  $r(\nu(k, j), i_{k+1}) = i_j$ . Thus

$$\zeta_{\nu(j-1, j)} = ((i_{j-1} + 1, i_j) - C'_{j-1, j} w'_{j-1}) \zeta_{\nu(j, j)}$$

(unless  $i_j = i_{j-1} + 1$  when  $\zeta_{\nu(j-1, j)} = \zeta_{\nu(j, j)}$ ) and

$$\zeta_{\nu(k, j)} = ((i_k + 1, i_{k+1} + 1) - C'_{kj} w'_k) \zeta_{\nu(k+1, j)}$$

for  $0 \leq k \leq j-2$ . By Theorem 2  $\mathcal{D}_m \zeta_{\mu(j, M)} = \theta_m^{-1} \zeta_{\tilde{\mu}(j, M)}$  where  $\tilde{\mu}(j, M) = \nu(0, j)$  (that is, first the  $i_j$ -entry of  $\lambda$  is moved to the  $m$ -entry at the end, the action of  $\mathcal{D}_m$  decrements  $\lambda_{i_j}$  by 1 and moves it to the front, loosely speaking). In turn  $\zeta_{\tilde{\mu}(j, M)}$  can be expressed in terms of  $\zeta_{\nu(j, j)}$ . The following is now established.

**Theorem 2.** Suppose  $\lambda \in \mathbb{N}_0^{N, P}$  with points of increase  $i_1 < i_2 < \dots < i_M = \ell(\lambda)$ , let  $R$  be the  $\mathbb{Z}$ -ring generated by  $\{C_{jk} : 1 \leq j < k \leq M\} \cup \{C'_{kj} : 0 \leq k < j \leq M\} \cup \mathbb{Z}$  and let  $\lambda^{(j)} = \lambda - \varepsilon(i_j) \in \mathbb{N}_0^{N, P}$ , then for  $i_{j-1} < i \leq i_j$  with  $0 \leq j \leq M$ ,

$$\mathcal{D}_i \zeta_\lambda = \sum_{s=j}^M ((N + 1 - i_s) \kappa + \lambda_{i_s}) u_{is} \zeta_{\lambda^{(s)}},$$

where each  $u_{is} \in RS[1, m]$ .

The Theorem exhibits the poles in the differentiation formula for the  $p$ -monic polynomials. To convert this for use with  $x$ -monic polynomials multiply  $\zeta_{\lambda^{(s)}}^x$  by  $(h(\lambda, 1) h(\lambda^{(s)}, \kappa + 1)) / (h(\lambda^{(s)}, 1) h(\lambda, \kappa + 1))$ , then the identity holds for  $\zeta$  replaced by  $\zeta^x$ . The details are not worked out since in general there is no significant simplification. In the next section this calculation will be carried out for the singular polynomials.

### 3. EXISTENCE OF SINGULAR POLYNOMIALS

In this section we will show for certain  $\lambda \in \mathbb{N}_0^{N,P}$  and singular values  $\kappa_0$  that  $\zeta_\lambda^x$  has no poles at  $\kappa = \kappa_0$  and that  $\mathcal{D}_i \zeta_\lambda^x = 0$  for  $1 \leq i \leq N$ . It turns out that for  $m = \ell(\lambda)$  the last coefficient in the formula of Theorem 2 satisfies  $(N + 1 - m) \kappa_0 + \lambda_m = 0$  and  $\zeta_{\lambda - \varepsilon(m)}$  has no poles at  $\kappa_0$  in general. For the terms of type  $\zeta_{\lambda^{(s)}}^x$  the denominator expression  $h(\lambda^{(s)}, \kappa + 1)$  has a zero at  $\kappa = \kappa_0$  but the pole  $(\kappa - \kappa_0)$  does not appear for the restriction to  $\mathbb{R}^N$ , and this is the key fact. We start with the isotypes of two-part partitions ( $\tau = (\mu, N - \mu)$ ).

**Proposition 4.** *Let  $\frac{N}{2} \leq \mu < N$ ,  $\gcd(m, \mu + 1) < \frac{\mu + 1}{N - \mu}$ ,  $\lambda = (m^{N - \mu})$  then  $h(\lambda, 1)$ ,  $h(\lambda, \kappa + 1)$ ,  $h(\lambda - \varepsilon(N - \mu), 1)$  and  $h(\lambda - \varepsilon(N - \mu), \kappa + 1)$  are nonzero when evaluated at  $\kappa = -\frac{m}{\mu + 1}$ .*

*Proof.* For  $1 \leq i \leq N - \mu$ ,  $1 \leq j \leq m$  we have  $h(\lambda, t; i, j) = m - j + t + (N - \mu - i) \kappa$ . For  $t = 1, \kappa + 1$  the sets of values are  $\{i\kappa + j : 0 \leq i \leq N - \mu - 1, 1 \leq j \leq m\}$ ,  $\{i\kappa + j : 1 \leq i \leq N - \mu, 1 \leq j \leq m\}$  respectively. It suffices to show that the second set does not contain 0 for  $\kappa = -\frac{m}{\mu + 1}$ . Suppose  $-im + j(\mu + 1) = 0$  for some (nonzero)  $i, j$  and let  $d = \gcd(m, \mu + 1)$ , then  $\frac{\mu + 1}{d} | i \leq N - \mu$  which implies  $\frac{\mu + 1}{N - \mu} \leq d$ , contrary to the hypothesis. For  $\lambda - \varepsilon(N - \mu)$  only the hook-lengths in the last row and column change;  $h(\lambda - \varepsilon(N - \mu), t; i, m) = t + (N - \mu - i) \kappa$  and  $h(\lambda - \varepsilon(N - \mu), t; N - \mu, j) = m - j + t$  for  $1 \leq i < N - \mu$  and  $1 \leq j < m$ . These values have already been shown to be nonzero for  $\kappa = -\frac{m}{\mu + 1}$ .  $\square$

Next we handle the case of three or more parts, for the isotype  $\tau = (s(\mu + 1) + \mu, \mu^l, \rho)$ . The following is the central hypothesis for this section.

**Definition 6.** *For  $\mu, l \geq 1, s \geq 0, 1 \leq \rho \leq \mu$  and  $\gcd(m, \mu + 1) = 1$  let*

$$\Lambda(\mu, s, l, \rho, m) = ((m(s + l + 1))^\rho, (m(s + l))^\mu, \dots, (m(s + 1))^\mu),$$

*a partition of length  $l\mu + \rho$  which is associated to the singular value  $\kappa_0 = -\frac{m}{\mu + 1}$  and the  $S_N$ -representation of isotype  $(s(\mu + 1) + \mu, \mu^l, \rho)$ , where  $N = (s + l + 1)\mu + s + \rho$ .*

**Lemma 4.** *Suppose  $a, b, c \in \mathbb{N}_0$  and  $c \geq 1, b \leq \mu$  then  $a(\mu\kappa + m) + b\kappa + c \neq 0$  for  $\kappa = -\frac{m}{\mu + 1}$  (where  $\gcd(m, \mu + 1) = 1$ ).*

*Proof.* Denote the value of the expression at  $\kappa = -\frac{m}{\mu + 1}$  by  $v$ , then  $v = (a - b) \frac{m}{\mu + 1} + c$ . If  $a \geq b$  then  $v \geq c \geq 1$ ; otherwise  $0 > a - b \geq -\mu$  and  $\mu + 1$  does not divide  $(a - b)m$  thus  $v \notin \mathbb{Z}$  and  $v \neq 0$ .  $\square$

**Proposition 5.** *Let  $\lambda = \Lambda(\mu, s, l, \rho, m)$ , then  $h(\lambda, 1)$  and  $h(\lambda, \kappa + 1)$  are nonzero when  $\kappa = -\frac{m}{\mu + 1}$ .*

*Proof.* Since hook-lengths in a given row depend only on it and the rows of higher index we may assume that  $\rho = \mu$ . We index the rows of  $\lambda$  by  $\mu k + i$  with  $0 \leq k \leq l$  and  $1 \leq i \leq \mu$ , and the columns by  $m(s+n) - j$  where  $1 \leq n \leq l+1-k$  and  $0 \leq j \leq m-1$  if  $n > 1$ , or  $0 \leq j < m(s+1)$  if  $n = 1$ . Then

$$\begin{aligned} h(\lambda, t; \mu k + i, m(s+n) - j) &= \kappa((l+2-k-n)\mu - i) + m(l+1-k-n) + j + t \\ &= (l+1-k-n)(\kappa\mu + m) + \kappa(\mu - i) + j + t. \end{aligned}$$

Set  $i' = \mu - i + 1$  if  $t = \kappa + 1$  or  $i' = \mu - i$  if  $t = 1$ ; then the above expression equals  $(l+1-k-n)(\kappa\mu + m) + \kappa i' + j + 1$  which is nonzero at  $\kappa = \kappa_0$  by Lemma 4 (since  $i' \leq \mu$ ).  $\square$

Next we consider the hook-lengths for  $\Lambda(\mu, s, l, \rho, m) - \varepsilon(\rho + k\mu)$  with  $0 \leq k \leq l$ .

**Proposition 6.** *For  $0 \leq k_0 \leq l$  let  $\nu = \Lambda(\mu, s, l, \rho, m) - \varepsilon(\rho + k_0\mu)$ , then for  $\kappa = -\frac{m}{\mu+1}$   $h(\nu, 1)$  is nonzero and  $h(\nu, \kappa + 1)$  is nonzero for  $k_0 = l$  and has a zero of multiplicity one for  $0 \leq k_0 < l$ , in the hook-length  $h(\nu, \kappa + 1; \rho + k_0\mu, m(s+l+1-k_0))$ .*

*Proof.* As in the previous proof, assume  $\rho = \mu$ . The column above the node deleted from  $\lambda$  (namely,  $m(s+l+1-k_0)$ ) meets the rows labeled by  $\mu k + i$  with  $0 \leq k \leq k_0$  and  $1 \leq i \leq \mu$ , except  $1 \leq i < \mu$  when  $k = k_0$ . Then  $h(\nu, t; \mu k + i, m(s+l+1-k_0)) = (k_0 - k)(\kappa\mu + m) + \kappa i' + 1$  where  $i' = \mu - 1 - i$  for  $t = 1$  and  $i' = \mu - i$  for  $t = \kappa + 1$ . By Lemma 4 the value is nonzero for  $\kappa = \kappa_0$ . The row of the deleted node meets the columns labeled  $m(s+n) - j$  with  $1 \leq n \leq l+1-k_0$ . Then  $h(\nu, t; m(s+n) - j, (k_0+1)\mu) = (l+1-k_0-n)(\kappa\mu + m) + b\kappa + j$ , where  $b = 0$  for  $t = 1$  and  $b = 1$  for  $t = \kappa + 1$ . The Lemma applies unless  $j = 0$ . Suppose  $j = 0$  then  $1 \leq n \leq l-k_0$  (the value  $j = 0$  does not occur for  $n = l+1-k_0$  since the corresponding node was deleted); at  $\kappa = \kappa_0$  the value of the hook-length is  $(l+1-k_0-n-b)\frac{m}{\mu+1}$  which is zero exactly when  $n = l-k_0$  and  $b = 1$  (that is,  $t = \kappa + 1$ ). Thus the hook-length  $h(\nu, \kappa + 1; m(s+l-k_0), (k_0+1)\mu) = \kappa(\mu + 1) + m$  is the only zero in  $h(\nu, \kappa + 1)$  at  $\kappa = \kappa_0$ .  $\square$

Next we show that the coefficients  $C_{jk}$  and  $C'_{kj}$  appearing in Theorem 2 have no poles at  $\kappa = \kappa_0$ . The points of decrease of  $\Lambda(\mu, s, l, \rho, m)$  are  $i_j = \rho + (j-1)\mu$ ,  $\lambda_{i_j} = m(s+2+l-j)$  for  $1 \leq j \leq l+1$ . For  $1 \leq j < k \leq l+1$  the coefficient  $C_{jk} = \frac{\kappa}{(i_k - i_j)\kappa + \lambda_{i_j} - \lambda_{i_k}} = \frac{\kappa}{(k-j)(\kappa\mu + m)}$ , which has value  $-\frac{1}{k-j}$  at  $\kappa = \kappa_0$ .

**Proposition 7.** *For  $\lambda = \Lambda(\mu, s, l, \rho, m)$  the coefficients  $C'_{kj}$  have no poles at  $\kappa = -\frac{m}{\mu+1}$  for  $0 \leq k < j \leq l+1$ .*

*Proof.* First the special cases  $C'_{j-1,j} = \frac{\kappa}{(\mu-1)\kappa+1}$  for  $j > 0, \mu > 1$  and  $C'_{0,1} = \frac{\kappa}{(\rho-1)\kappa+1}$  for  $\rho > 1$  are obviously finite at  $\kappa = \kappa_0$ . Next for  $j > 1$  we have  $C'_{0,j} = \frac{\kappa}{((j-1)\mu + \rho - 1)\kappa + m(j-1) + 1}$  with denominator  $(j-1)(\mu\kappa + m) + (\rho-1)\kappa + 1$  which is nonzero at  $\kappa = \kappa_0$  by Lemma 4, since  $\rho-1 \leq \mu$ . Finally for  $1 \leq k < j-1 \leq l$  we have  $C'_{k,j} = \kappa((j-k-1)(\mu\kappa + m) + (\mu-1)\kappa + 1)^{-1}$ , and the Lemma applies.  $\square$

We restate the result of Theorem 2 applied to the  $x$ -monic polynomials  $\zeta_\lambda^x$  and  $\zeta_{\lambda-\varepsilon(\rho+k\mu)}^x$  (for  $\lambda = \Lambda(\mu, s, l, \rho, m)$  and  $0 \leq k \leq l$ ). For  $1 \leq i \leq \rho + l\mu$

$$\begin{aligned} \mathcal{D}_i \zeta_\lambda^x &= \sum_{k=0}^l ((l-k)(\kappa\mu + m) + (s+1)(m + \kappa(\mu+1))) u_{i,k+1} \\ &\quad \times \frac{h(\lambda, 1) h(\lambda - \varepsilon(\rho + k\mu), \kappa + 1)}{h(\lambda, \kappa + 1) h(\lambda - \varepsilon(\rho + k\mu), 1)} \zeta_{\lambda - \varepsilon(\rho + k\mu)}^x, \end{aligned}$$

where (the labeling of the points of decrease is now shifted by 1) each  $u_{i,k+1} \in RS[1, \rho + l\mu]$  and  $R$  is the ring generated by  $\{C_{jk} : 1 \leq j < k \leq l+1\} \cup \{C'_{kj} : 0 \leq k < j \leq l+1\} \cup \mathbb{Z}$ ; also  $u_{i,k+1} = 0$  for  $k < \frac{i-\rho}{\mu}$ . Since  $h(\lambda, \kappa+1) \neq 0$  at  $\kappa = \kappa_0$  the polynomial  $\zeta_\lambda$  has no poles there. Also the specialization of  $R$  is a subring of  $\mathbb{Q}$ . For  $k = l$  we already have shown that  $h(\lambda - \varepsilon(\rho + l\mu), \kappa + 1) \neq 0$  and thus  $\zeta_{\lambda - \varepsilon(\rho + l\mu)}^x$  has no poles at  $\kappa_0$  and the factor  $(s+1)(m + \kappa(\mu+1))$  becomes zero. When  $0 \leq k < l$  the factor  $h(\lambda - \varepsilon(\rho + k\mu), \kappa + 1)$  has a zero at  $\kappa_0$ . Once we prove that  $\zeta_{\lambda - \varepsilon(\rho + k\mu)}^x$  has no pole at  $\kappa_0$  the proof that  $\mathcal{D}_i \zeta_\lambda = 0$  for all  $i$  will be complete.

The method of Knop and Sahi [11] was designed to show that the coefficients of the monomials  $x^\beta$  in  $h(\lambda, \kappa + 1) \zeta_\lambda^x$  are in  $\mathbb{N}_0[\kappa]$ , but it is not evident how to use the method to identify the poles when the number of variables is in the range  $\ell(\lambda) \leq N < \ell(\lambda) + |\lambda|$ . We introduce a different approach.

**Definition 7.** Let  $\alpha, \beta \in \mathbb{N}_0^M$  with  $\alpha \triangleright \beta$  and let  $m, n \in \mathbb{N}$  with  $\gcd(m, n) = 1$  then say  $(\alpha, \beta)$  is a  $(-\frac{m}{n})$ -critical pair if  $(n\kappa + m)$  divides  $(r(\beta, i) - r(\alpha, i))\kappa + \alpha_i - \beta_i$  (in  $\mathbb{Q}[\kappa]$ ) for  $1 \leq i \leq M$ .

In fact the division is in  $\mathbb{Z}[\kappa]$  because  $\gcd(m, n) = 1$ . The definition will be used in the situation  $\alpha \in \mathbb{N}_0^{N,P}$  that is,  $\ell(\alpha) \leq N$  and  $M = \ell(\alpha) + |\alpha|$ . See Definitions 2 and 3 for the order  $\triangleright$  and the rank function  $r$ .

**Theorem 3.** Suppose  $\lambda \in \mathbb{N}_0^{N,P}$  and  $\kappa_0 \in \mathbb{Q}, \kappa_0 < 0$ ; if there does not exist  $\beta \in \mathbb{N}_0^N$  such that  $(\lambda, \beta)$  is a  $\kappa_0$ -critical pair then  $\kappa_0$  is not a pole of  $\zeta_\lambda^x$  restricted to  $\mathbb{R}^N$ .

*Proof.* Extend the field  $\mathbb{Q}(\kappa)$  with  $N$  transcendental variables  $\{v_1, v_2, \dots, v_N\}$  and let  $\mathcal{T} = \sum_{i=1}^N v_i \mathcal{U}_i$ . For each  $\alpha \in \mathbb{N}_0^N$  the polynomial  $\zeta_\alpha^x$  is an eigenvector of  $\mathcal{T}$ , indeed  $\mathcal{T} \zeta_\alpha^x = \sum_{i=1}^N v_i \xi_i(\alpha) \zeta_\alpha^x$ . The eigenvalue determines  $\alpha$  uniquely for generic  $\kappa$  (with the possible exception of a finite set of negative rationals). Let  $C = \{\beta \in \mathbb{N}_0^N : \lambda \triangleright \beta\}$ . By the triangularity of the operators  $\{\mathcal{U}_i\}$  we have  $x^\lambda = \zeta_\lambda^x + \sum_{\beta \in C} B_{\beta\lambda} \zeta_\beta^x$  for certain coefficients  $B_{\beta\lambda} \in \mathbb{Q}(\kappa)$ . Let

$$\mathcal{T}_\lambda = \prod_{\beta \in C} \frac{\mathcal{T} - \sum_{i=1}^N v_i \xi_i(\beta)}{\sum_{i=1}^N v_i (\xi_i(\lambda) - \xi_i(\beta))},$$

then  $\mathcal{T}_\lambda x^\lambda = \zeta_\lambda^x$  (note that the number  $N$  of variables is part of the definition of the set  $C$ ). The numerator of the product is a polynomial in  $\kappa, v_1, \dots, v_N$  (and of course each  $\mathcal{D}_i x^\alpha$  is a polynomial with coefficients in  $\mathbb{Z}[\kappa]$ ) thus any  $(\kappa)$ -poles in  $\zeta_\lambda^x$  must appear in the set  $\{\sum_{i=1}^N v_i (\xi_i(\lambda) - \xi_i(\beta)) : \lambda \triangleright \beta\}$ . For any  $\beta \in C$  we have  $\sum_{i=1}^N v_i (\xi_i(\lambda) - \xi_i(\beta)) = \sum_{i=1}^N v_i ((r(\beta, i) - r(\lambda, i))\kappa + \lambda_i - \beta_i)$ . Since any denominator appearing in a coefficient (with respect to the  $x$ -monomial basis) of  $\zeta_\lambda^x$  must be a factor of  $h(\lambda, \kappa + 1)$ , all of the terms involving  $\{v_i : 1 \leq i \leq N\}$

must cancel out in the calculation of  $\mathcal{T}_\lambda x^\lambda$ . Thus the irreducible polynomials  $\sum_{i=1}^N v_i (\xi_i(\lambda) - \xi_i(\beta))$  must cancel out and the denominators in  $\zeta_\lambda^x$  can only arise from reducible terms of the form  $\left(\sum_{i=1}^N a_i v_i\right) (\kappa - \kappa_1)$  where  $a_1, \dots, a_N, \kappa_1 \in \mathbb{Q}$ . This condition is equivalent to  $(\lambda, \beta)$  being a  $\kappa_1$ -critical pair. Thus, if there is no  $\kappa_0$ -critical pair  $(\lambda, \beta)$  with  $\ell(\beta) \leq N$  then  $\kappa_0$  is not a pole of  $\zeta_\lambda^x$ .  $\square$

We will exploit this theorem by directly constructing the unique  $\beta$  such that  $(\Lambda(\mu, s, l, \rho, m) - \varepsilon(\rho + k\mu), \beta)$  is  $-\frac{m}{\mu+1}$ -critical. Here is a numerical example: for  $N = 33$  consider  $\Lambda(3, 4, 4, 2, 3)$  for the singular value  $\kappa_0 = -\frac{3}{4}$  of isotype  $(19, 3^4, 2)$ , take  $k = 1$ , then  $\lambda = (27^2, 24^2, 23, 21^3, 18^3, 15^3)$  and the unique  $\beta$  such that  $(\lambda, \beta)$  is  $(-\frac{3}{4})$ -critical is  $(27^2, 24^2, 2, 0^3, 21^3, 18^3, 3^{22})$ . The construction proceeds through several lemmas. Fix  $k$  such that  $0 \leq k \leq l-1$ , let  $\lambda = \Lambda(\mu, s, l, \rho, m) - \varepsilon(\rho + k\mu)$ ,  $L = \ell(\lambda) = \rho + l\mu$ ; and partition  $[1, L]$  into cells  $\{I_j : 0 \leq j \leq l\}$ , where  $I_0 = [1, \rho]$  and  $I_j = [\rho + (j-1)\mu + 1, \rho + j\mu]$  for  $1 \leq j \leq l$ . Then  $i \in I_j$  implies  $\lambda_i = m(l + s + 1 - j)$ , except that  $\lambda_{\rho+k\mu} = m(l + s + 1 - k) - 1$ . We will show the required  $\beta$  has the values  $m(l + s + 1 - j)$  on the cells  $I_j$  with  $j \leq k$ , 0 on  $I_{k+1}$ ,  $m(l + s + 2 - j)$  on  $I_j$  with  $k < j \leq l$ , except  $\beta_{\rho+k\mu} = m - 1$ , and  $\beta_i = m$  for  $L + 1 \leq i \leq L + \mu(s + 1) + l + s - k = N + l - k$ ; also  $\ell(\beta) = N + l - k$ . Henceforth, suppose that  $(\lambda, \beta)$  is  $-\frac{m}{\mu+1}$ -critical or  $\beta = \lambda$ . This holds if and only if the *rank equation*

$$(3.1) \quad r(\beta, i) - i = (\mu + 1) \left( \frac{1}{m} (\lambda_i - \beta_i) \right)$$

is satisfied for all  $i \geq 1$ . Since  $\gcd(m, \mu + 1) = 1$  this implies that  $\beta_i \equiv \lambda_i \pmod{m}$  (so with the exception of  $\beta_{\rho+k\mu}$  each  $\beta_i$  is divisible by  $m$ ). Here is a maximum principle for the multiplicity  $\#\{j : \beta_j = m\gamma, 1 \leq j \leq L\}$  for any  $\gamma$ . There is a slight difference for the cases  $m = 1$  and  $m > 1$ . The condition  $\lambda \succeq \beta$  implies that any possible values satisfy  $\gamma \leq s + l + 1$ .

**Lemma 5.** *Suppose  $\gamma \in \mathbb{N}_0$ , and  $G = \{j : \beta_j = m\gamma, 1 \leq j \leq L\}$ ,  $m > 1$  or  $m = 1$  and  $\rho + k\mu \notin G$ , if  $G$  meets two or more cells then  $\#G \leq \mu - 1$ ; additionally, if one of the cells is  $I_0$  then  $\#G \leq \rho - 1$ .*

*Proof.* Let  $G$  have nonempty intersections with cells  $I_{g_1}, I_{g_2}, \dots, I_{g_u}$  with  $0 \leq g_1 < g_2 < \dots < g_u \leq l$ . By hypothesis  $\rho + k\mu \notin G$  (if  $m > 1$  then  $m$  does not divide  $\beta_{\rho+k\mu}$ ) and so  $i \in G \cap I_{g_a}$  implies  $\lambda_i = m(s + l + 1 - g_a)$ . Each  $G \cap I_{g_a}$  is an interval  $[i_a, j_a]$ ; indeed suppose  $i, j \in G \cap I_{g_a}$  and  $i < j$ , then by equation (3.1)  $r(\beta, i) - i = (\mu + 1) \left( \frac{1}{m} (m(s + l + 1 - g_a) - m\gamma) \right) = r(\beta, j) - j$ ; thus  $r(\beta, j) = r(\beta, i) + j - i$ . Since  $\beta_j = \beta_i$  this implies that  $\beta_b = \beta_i = m\gamma$  for  $i \leq b \leq j$ . For  $0 \leq a < u$  we have that  $r(\beta, i_{a+1}) = r(\beta, j_a) + 1$  and we combine the two equations

$$\begin{aligned} r(\beta, i_{a+1}) - i_{a+1} &= (\mu + 1)(l + s + 1 - g_{a+1} - \gamma), \\ r(\beta, j_a) - j_a &= (\mu + 1)(l + s + 1 - g_a - \gamma) \end{aligned}$$

to obtain  $i_{a+1} = j_a + 1 + (\mu + 1)(g_{a+1} - g_a)$ . Then  $\#G = \sum_{a=1}^u (j_a - i_a + 1) = u + j_u - i_1 - \sum_{a=1}^{u-1} (i_{a+1} - j_a) = u + j_u - i_1 - \sum_{a=1}^{u-1} (1 + (\mu + 1)(g_{a+1} - g_a)) = 1 + j_u - i_1 - (\mu + 1)(g_u - g_1)$ . But  $j_u \leq \rho + g_u\mu$  and  $i_1 \geq \rho + (g_1 - 1)\mu + 1$  for  $g_1 \geq 1$  while  $i_1 \geq 1$  for  $g_1 = 0$ . This shows that  $\#G \leq \mu - (g_u - g_1)$  for  $g_1 \geq 1$  and  $\#G \leq \rho - g_u$  if  $g_1 = 0$ . In both cases  $\#G \leq \mu - 1$ .  $\square$

**Lemma 6.** *Suppose  $m = 1, \gamma \in \mathbb{N}_0, G = \{j : \beta_j = \gamma, 1 \leq j \leq L\}$ , and  $G \setminus \{\rho + k\mu\}$  meets two or more cells, if  $\rho + k\mu = \min(G)$  and  $G \neq [\rho + k\mu, \rho + (k+1)\mu]$  then  $\#G \leq \mu$ , otherwise ( $\rho + k\mu \neq \min(G)$ ) then  $\#G \leq \mu - 1$ .*

*Proof.* By hypothesis  $G \neq [\rho + k\mu, \rho + (k+1)\mu]$ . We can apply the previous argument if we replace  $I_k$  by  $I_k \setminus \{\rho + k\mu\}$  and  $I_{k+1}$  by  $I_{k+1} \cup \{\rho + k\mu\} = [\rho + k\mu, \rho + (k+1)\mu]$ . If  $g_1 \neq k+1$  then as before  $\#G \leq \mu - (g_u - g_1) \leq \mu - 1$ . If  $g_1 = k+1$  and  $i_1 \geq \rho + k\mu + 1$  the same conclusion results. When  $g_1 = k+1$  and  $i_1 = \rho + k\mu$ , that is,  $\min G = \rho + k\mu$ , the calculation yields the bound  $\#G \leq \mu + 1 - (g_u - k - 1) \leq \mu$ .  $\square$

The two lemmas show that  $\#\{j : \beta_j = \gamma m, 1 \leq j \leq L\} \leq \mu$  for any  $\gamma \in \mathbb{N}_0$ , except when  $m = 1$  and  $\{j : \beta_j = \gamma\} = [\rho + k\mu, \rho + (k+1)\mu]$  of cardinality  $\mu + 1$ . Next we show  $\beta_{L+1} \leq m$ .

**Lemma 7.** *Either  $\beta_{L+1} = m$  and  $r(\beta, L+1) = L - \mu$ , or  $\beta_{L+1} = 0$  and  $r(\beta, L+1) = L + 1, \ell(\beta) = L$ .*

*Proof.* Denote  $\frac{\beta_{L+1}}{m}$  by  $b$ ; note that  $b \in \mathbb{N}_0$ . First we show  $b \leq l$ : by equation 3.1  $r(\beta, L+1) = L + 1 - (\mu + 1)b \geq 1$  and so  $b \leq \frac{L+1}{\mu+1} \leq (l+1) \frac{\mu}{\mu+1} < l+1$ . Let  $a_0 = \#\{j : 1 \leq j \leq L, \beta_j < \beta_{L+1}\}$  and  $a_1 = \#\{j : j > L, \beta_j > \beta_{L+1}\}$  then  $r(\beta, L+1) = L + 1 - a_0 + a_1 \geq L + 1 - a_0$ . We claim  $a_0 \leq b\mu + 1$ . If  $m > 1$  then  $a_0 = \sum_{i=0}^{b-1} \#\{j : \beta_j = im, 1 \leq j \leq L\} + \#\{j : \beta_j = cm - 1, c \leq b\}$ . By the maximum principle  $a_0 \leq b\mu + 1$ . If  $m = 1$  then  $a_0 = \sum_{i=0}^{b-1} \#\{j : \beta_j = i, 1 \leq j \leq L\}$ ; at most one of these sets can have cardinality  $\mu + 1$  and again  $a_0 \leq b\mu + 1$ . Then  $L + 1 - (\mu + 1)b = r(\beta, L+1) \geq L - b\mu$ , that is,  $b \leq 1$ . If  $b = 1$  then  $r(\beta, L+1) = L - \mu$ . If  $b = 0$  then  $r(\beta, L+1) = L + 1$  which implies  $\beta_j = 0$  for all  $j > L$ . The hypothesis  $\lambda \succeq \beta$  implies  $L = \ell(\lambda) \leq \ell(\beta)$ .  $\square$

In fact,  $\beta_{L+1} = 0$  implies  $\beta = \lambda$  and  $\beta_{L+1} = m$  corresponds to a unique solution  $\beta$  with  $\ell(\beta) = N + l - k$ .

**Lemma 8.** *Suppose that  $\beta_{L+1} = m$  then  $\beta_{\rho+k\mu} = m - 1, \beta_i = 0$  for  $i \in I_{k+1}, \beta_i = m$  for  $L + 1 \leq i \leq N + l - k$  and  $\ell(\beta) = N + l - k$ .*

*Proof.* Let  $a_0 = \#\{j : 1 \leq j \leq L, \beta_j \geq m\}$ ,  $a_1 = \#\{j : L < j, \beta_j > m\}$ ,  $G_0 = \{j : 1 \leq j \leq L, \beta_j = 0\}$ ,  $G_1 = \{j : \beta_j = m - 1 > 0\}$ , and  $a_2 = \#G_0 + \#G_1$  where  $G_1$  is empty when  $m = 1$ . Then  $L - \mu = r(\beta, L+1) = a_0 + a_1 + 1$  and  $L = a_0 + a_2$ , thus  $a_2 = \mu + 1 + a_1 \geq \mu + 1$ . But by the maximum principle  $a_2 \leq \mu + 1$ , hence  $a_1 = 0$  and  $a_2 = \mu + 1$ . If  $m > 1$  then  $\#G_0 \leq \mu$  implying that  $\#G_1 = 1$  and  $G_1 = \{\rho + k\mu\}$ , also  $\#G_0 = \mu$  and thus  $G_0 = I_j$  for some  $j \neq 0, k$  by Lemma 5. If  $m = 1$  then  $G_0 = [\rho + k\mu, \rho + (k+1)\mu]$  by Lemma 6. Let  $r_0 = r(\beta, \rho + k\mu)$ , then  $r(\beta, \rho + (j-1)\mu + 1) = r_0 + 1$ ; if  $m = 1$  then  $\beta_{\rho+k\mu} = \beta_{\rho+k\mu+1} = 0$  and  $j = k+1$ , while for  $m > 1$  we have  $\beta_{\rho+k\mu} = m - 1$  which is the unique minimum of  $\{\beta_j : 1 \leq j \leq L, \beta_j > 0\}$  and  $\beta_{\rho+(j-1)\mu+1}$  is the first occurrence of 0. By equation 3.1

$$\begin{aligned} r_0 - (\rho + k\mu) &= \frac{\mu + 1}{m} (m(s + l + 1 - k) - 1 - (m - 1)) \\ &= (\mu + 1)(s + l - k), \\ r_0 + 1 - (\rho + (j-1)\mu + 1) &= (\mu + 1)(s + l + 1 - j). \end{aligned}$$

Thus  $r_0 = \rho + k\mu + (\mu + 1)(s + l - k)$  and  $(j - 1 - k)\mu = (\mu + 1)(j - 1 - k)$ , that is  $j = k + 1$ . But  $r_0 = \#\{j : \beta_j \geq m\} + 1 = L - \mu + \#\{j : L < j, \beta_j = m\}$  and so  $\#\{j : L < j, \beta_j = m\} = \mu s + \mu + s + l - k = N - L + l - k$ . This shows  $\ell(\beta) = N + l - k \geq N + 1$ .  $\square$

Certainly this, together with a proof that  $\ell(\beta) = L$  implies  $\beta = \lambda$ , is enough for the main purpose, but with not much more work we can show that  $\beta$  is unique. In fact we will show that  $\beta_{L+1} = m$  implies that for  $i \in I_j$   $\beta_i = m(s + l + 1 - j)$  for  $j \leq k$  and  $\beta_i = m(s + l + 2 - j)$  for  $j > k + 1$ , except  $\beta_{\rho+k\mu} = m - 1$ .

**Lemma 9.** *Suppose that  $\beta_{L+1} = m$ , then  $\beta_i = \lambda_i$  for all  $i < \rho + k\mu$  and  $\beta_i = \lambda_i + m$  for  $\rho + (k + 1)\mu + 1 \leq i \leq \ell(\beta)$ .*

*Proof.* For  $0 \leq i \leq s + l$  let  $M_i = \#\{j : 1 \leq j \leq L, \beta_j = m(s + l + 1 - i)\}$ . Since  $\lambda_i = m(s + l + 1)$  for  $1 \leq i \leq \rho$  when  $k \geq 1$ , and for  $1 \leq i \leq \rho - 1$  when  $k = 0$ , the condition  $\lambda \triangleright \beta$  (thus  $\lambda \succeq \beta^+$ ) implies  $M_0 \leq \rho$  or  $\rho - 1$  respectively. The maximum principle (Lemmas 5 and 6) implies that  $M_i \leq \mu$  for  $1 \leq i < l + 1$  (from the previous lemma in which  $\beta_{\rho+k\mu} = m - 1$  was determined). Further  $\sum_{i=0}^{s+l} M_i = L - \mu - 1 = \rho + (l - 1)\mu - 1$ , that is,  $\sum_{i=l}^{s+l} M_i = \rho - M_0 + \sum_{i=1}^{l-1} (\mu - M_i) - 1$ . Also

$$\begin{aligned} |\beta| &= \sum_{i=0}^{s+l} M_i m(s + l + 1 - i) + m - 1 + m(\mu s + \mu + s + l - k) \\ &= \rho m(s + l + 1) + m\mu \sum_{i=1}^l (s + i) - 1, \end{aligned}$$

and so

$$\begin{aligned} \sum_{i=l}^{s+l} M_i (s + l + 1 - i) &= (\rho - M_0)(s + l + 1) \\ &\quad + \sum_{i=1}^{l-1} (\mu - M_i)(s + l + 1 - i) - (s + l + 1 - k). \end{aligned}$$

Let  $j$  be defined by  $M_0 = \rho$ ,  $M_i = \mu$  for  $1 \leq i \leq j$  and  $M_{j+1} \leq \mu - 1$ , that is,  $j \geq 0$ , while  $j = -1$  when  $M_0 \leq \rho - 1$ . The hypothesis  $\lambda \succeq \beta^+$  implies  $j < k$  (or else  $\sum_{i=1}^{\rho+k\mu} \beta_i^+ > \sum_{i=1}^{\rho+k\mu} \lambda_i$ ). Then for  $j = -1$  we have

$$\begin{aligned} \sum_{i=l}^{s+l} M_i (s + l + 1 - i) &= (\rho - 1 - M_0)(s + l + 1) + \sum_{i=1}^{l-1} (\mu - M_i)(s + l + 1 - i) + k \\ &\geq (s + 2) \left( \rho - 1 - M_0 + \sum_{i=1}^{l-1} (\mu - M_i) \right) = \sum_{i=l}^{s+l} M_i (s + 2), \end{aligned}$$

and for  $j \geq 0$

$$\begin{aligned} \sum_{i=l}^{s+l} M_i (s + l + 1 - i) &= (\rho - M_0)(s + l + 1) + \sum_{i=1, i \neq j+1}^{l-1} (\mu - M_i)(s + l + 1 - i) \\ &\quad + (\mu - 1 - M_{j+1})(s + l - j) + (k - 1 - j) \\ &\geq \sum_{i=l}^{s+l} M_i (s + 2), \end{aligned}$$

since each coefficient is nonnegative, but then  $M_i = 0$  for all  $i \geq l$ . The nonnegativity of each term on the right hand sides implies  $j = k - 1$  and if  $k = 0$  then  $M_0 = \rho - 1$  and  $M_i = \mu$  for  $1 \leq i \leq l - 1$ , or else  $M_0 = \rho$ ,  $M_k = \mu - 1$  and  $M_i = \mu$  for  $1 \leq i \leq l - 1, i \neq k$ . Let  $G_i = \{j : 1 \leq j \leq L, \beta_j = m(s + l + 1 - i)\}$  for  $2 \leq i \leq l + 1$ . By Lemma 5 for each  $i$  satisfying  $1 \leq i \leq l - 1, i \neq k$  there is  $u_i$  such that  $G_i = I_{u_i}$  and  $u_i \neq k, k + 1$ .

If  $k = 0$  then  $G_0 = [1, \rho - 1]$  since all other cells are of cardinality  $\mu$ . For each  $G_i$  with  $i \geq 1$  the rank of the first coordinate is  $\rho + (i - 1)\mu$ , that is  $r(\beta, \rho + (u_i - 1)\mu + 1) = \rho + (i - 1)\mu$ . Then by equation 3.1

$$\rho + (i - 1)\mu - (\rho + (u_i - 1)\mu + 1) = (\mu + 1)((s + l + 1 - u_i) - (s + l + 1 - i)),$$

thus  $u_i = i + 1$ , for  $1 \leq i \leq l - 1$ . If  $k > 0$  we have shown  $\#G_0 = \rho$  and  $\#G_k = \mu - 1$ . If  $\rho \leq \mu - 1$  then neither  $G_0$  nor  $G_k$  can meet  $I_0$  and another cell, by Lemma 5. If additionally  $\rho < \mu - 1$  then  $G_0 = I_0$  and  $G_k = [\rho + (k - 1)\mu + 1, \rho + k\mu - 1]$ . If  $\rho = \mu - 1$  then it is not possible for  $G_k = I_0$  because then  $r(\beta, 1) = \rho + (k - 1)\mu + 1$  and equation 3.1 yields  $\rho + (k - 1)\mu = (\mu + 1)k$ , that is  $k + 1 = 0$ . As before,  $G_0 = I_0$  and  $G_k = I_k \setminus \{\rho + k\mu\}$ . If  $\rho = \mu$  then  $G_k = I_k \setminus \{\rho + k\mu\}$  and  $G_0 = I_{u_0}$  for some  $u_0$ . The needed ranks for  $\beta$  are  $r(\beta, \rho + (u_i - 1)\mu + 1) = \rho + (i - 1)\mu + 1$  if  $i < k$  and  $= \rho + (i - 1)\mu$  if  $k < i \leq l - 1$ . Similarly to the case  $k = 0$  this implies that  $u_i = i$  for  $i < k$  and  $u_i = i + 1$  for  $k < i \leq l - 1$ .  $\square$

It remains to show that  $\ell(\beta) \leq L$  implies  $\beta = \lambda$ .

**Lemma 10.** *Suppose  $\beta_{L+1} = 0$ , then  $\beta = \lambda$ .*

*Proof.* The hypothesis implies  $\beta_i = 0$  for all  $i > L$  (the rank equation showed  $r(\beta, L + 1) = L + 1$  thus  $i > L + 1$  implies  $\beta_i = 0$ ). The condition  $\lambda \succeq \beta^+$  implies  $\beta_L^+ \geq \lambda_L = m(s + 1)$  (since  $|\lambda| - \lambda_L \geq |\beta| - \beta_L^+$ ). For  $1 \leq i \leq l + 1$  let  $G_i = \{j : \beta_j = m(s + l + 1 - i)\}$  and  $M_i = \#G_i$ . Firstly let  $m > 1$ , then  $\beta_{\rho+k\mu} = m(s + l + 1 - j_0) - 1$  for some  $j_0$  in  $0 \leq j \leq l - 1$ , thus  $\rho + l\mu = \sum_{i=0}^l M_i + 1$  and  $\rho - M_0 + \sum_{i=1}^l (\mu - M_i) = 1$ . Also  $M_0 \leq \rho$  because  $\lambda \succeq \beta^+$  and  $M_i \leq \mu$  for  $1 \leq i \leq l$  by Lemma 5. Hence either  $M_0 = \rho - 1$  and  $M_i = \mu$  for  $1 \leq i \leq l$ , or for some  $j > 0$   $M_j = \mu - 1$  and  $M_0 = \rho$ ,  $M_i = \mu$  for  $1 \leq i \leq l, i \neq j$ . Now

$$\begin{aligned} |\beta| &= \rho m(s + l + 1) + \mu m \sum_{i=1}^l (s + l + 1 - i) - 1 \\ &= M_0 m(s + l + 1) + m \sum_{i=1}^l M_i (s + l + 1 - i) + m(s + l + 1 - j_0) - 1, \end{aligned}$$

and substituting the known values for  $M_i$  we obtain  $j_0 = 0$  if  $M_0 = \rho - 1$  else  $j_0 = j$ . Then  $r(\beta, \rho + k\mu) = \rho + j_0\mu$  and the rank equation at  $\rho + k\mu$  yields  $(j_0 - k)\mu = (\mu + 1)(j_0 - k)$  and so  $j_0 = k$ , that is,  $\beta_{\rho+k\mu} = \lambda_{\rho+k\mu}$ . Similarly to the previous lemma let  $G_i = \{j : \beta_j = m(s + l + 1 - i)\} = I_{u_i}$  for  $0 \leq i \leq l, i \neq k$  and some  $u_i \neq k$ , treating the special cases  $\rho < \mu - 1$ ,  $\rho = \mu - 1$  and  $\rho = \mu$  as before. Again  $r(\beta, \rho + (i - 1)\mu + 1) = \rho + (u_i - 1)\mu + 1$  and the rank equation shows  $u_i = i$ . Also  $G_k = I_k \setminus \{\rho + k\mu\}$ . Thus  $\beta = \lambda$ .

Secondly let  $m = 1$ . Then  $\lambda_i = s + l - k$  for  $\rho + k\mu \leq i \leq \rho + (k+1)\mu$ . Also  $\rho \geq M_0$  because  $\lambda \succeq \beta^+$ . There are two equations involving  $M_i$ :

$$(3.2) \quad (\rho - M_0) + \sum_{i=1}^l (\mu - M_i) = 0,$$

$$(3.3) \quad (\rho - M_0)(s + l + 1) + \sum_{i=1}^l (\mu - M_i)(s + l + 1 - i) = 1.$$

Equation 3.3 shows that  $\rho \geq M_0$  and  $\mu \geq M_i$  for all  $i$  is impossible, hence there is at least one value, say  $M_j$ , such that  $M_j > \mu$ . By the maximum principle  $M_j = \mu + 1$  and  $M_i \leq \mu$  for all  $i \neq j$ . Substituting these conditions in equation 3.2 shows that for some  $j_0$ ,  $M_{j_0} = \mu - 1$  ( $M_0 = \rho - 1$  if  $j_0 = 0$ ) and  $M_i = \mu$  for all  $i \neq j_0, j$ , and  $M_0 = \rho$  unless  $j_0 = 0$ . Substitute these values in equation 3.3 to obtain  $j_0 = j - 1$ . By Lemma 6  $G_j = [\rho + k\mu, \rho + (k+1)\mu]$ , also  $r(\beta, \rho + k\mu) = \rho + (j-1)\mu$ . Then the rank equation shows  $(j-1-k)\mu = (\mu+1)((s+l-k) - (s+l+1-j)) = (\mu+1)(j-1-k)$  and thus  $j = k+1$ . Similarly to the previous arguments, for each  $i \neq k, k+1$  there exist  $u_i$  such that  $G_i = I_{u_i}$ . Since  $M_k + M_{k+1} = 2\mu$  (or  $\rho + \mu$  if  $k = 0$ ) we have  $r(\beta, \rho + (u_i - 1)\mu + 1) = \rho + (i-1)\mu + 1$  and the rank equation shows  $u_i = i$ . This accounts for all of  $[1, L]$  except for  $[1, \rho]$  and  $[\rho + (k-1)\mu + 1, \rho + k\mu - 1]$ . There are several cases for  $\rho$ : if  $k = 0$  then  $G_0 = [1, \rho - 1]$  by elimination; if  $k \geq 1$  and  $\rho = \mu$  then  $G_0 = I_{u_0}$  and the rank equation shows  $u_0 = 0$ , and  $G_k = [\rho + (k-1)\mu + 1, \rho + k\mu - 1]$ ; if  $k \geq 1$  and  $\rho = \mu - 1$  then by Lemma 6  $G_0$  can not meet both  $I_0$  and  $I_k$  thus either  $G_0 = I_0$  or  $G_0 = I_k \setminus \{\rho + k\mu\}$  and the rank equation implies the latter can not happen; if  $k \geq 1$  and  $\rho < \mu - 1$  then by the same Lemma  $G_k = I_k \setminus \{\rho + k\mu\}$ , forcing  $G_0 = I_0$ . Thus  $\beta = \lambda$ .  $\square$

The lemmas together provide the proofs of the following theorems.

**Theorem 4.** *Let  $\lambda = \Lambda(\mu, s, l, \rho, m)$  and  $\lambda^{(k)} = \lambda - \varepsilon(\rho + k\mu)$  for  $0 \leq k \leq l-1$ . Then there exists a unique  $\beta$  so that  $(\lambda^{(k)}, \beta)$  is  $\left(-\frac{m}{\mu+1}\right)$ -critical and  $\ell(\beta) = N + l - k > N$ , where  $N = (s + l + 1)\mu + s + \rho$ .*

**Theorem 5.** *Let  $\lambda = \Lambda(\mu, s, l, \rho, m)$  and  $N = (s + l + 1)\mu + s + \rho$  then  $\zeta_\lambda^x$  is a singular polynomial for  $S_N$  with singular value  $-\frac{m}{\mu+1}$ .*

In the next section we study the irreducible representation associated to  $\zeta_\lambda^x$ , in particular, an explicit basis for the span of its  $S_N$ -orbit.

#### 4. ASSOCIATED $S_N$ -MODULES

Using Murphy's construction [13] of Young's seminormal representations we can give a complete description of the  $S_N$ -orbit of  $\zeta_\lambda^x$ . From the formula (valid for all  $\kappa$  and for all polynomials  $f$ )

$$\sum_{i=1}^N x_i \mathcal{D}_i f(x) = \sum_{i=1}^N x_i \frac{\partial}{\partial x_i} f(x) + \kappa \sum_{1 \leq i < j \leq N} (f(x) - f(x(i, j)))$$

we note that a homogeneous singular polynomial  $f$  must satisfy  $(\deg f) f = -\kappa \omega f$  where  $\omega = \sum_{1 \leq i < j \leq N} (1 - (i, j))$ . But  $\omega$  is in the center of  $\mathbb{Z}S_N$  and the eigenvalues for any isotype are known (Young's formula). Indeed for any node  $(i, j)$  in the

Ferrers diagram of a partition  $\tau$  (with  $|\tau| = N$ ), the content is defined to be  $c((i, j)) = j - i$ , then  $\omega f = \left( \binom{N}{2} - \sum_{(i, j) \in \tau} c((i, j)) \right) f$  whenever  $f$  is of isotype  $\tau$ . Denote the eigenvalue by  $\tau(\omega)$ , then  $\tau(\omega) = \binom{N}{2} - \frac{1}{2} \sum_{i=1}^{\ell(\tau)} \tau_i (\tau_i + 1 - 2i)$ . As a function on partitions the eigenvalue is strictly decreasing with respect to the dominance order.

**Lemma 11.** *Suppose  $\sigma, \tau \in \mathbb{N}_0^{N, P}$ ,  $|\sigma| = |\tau|$  and  $\sigma \prec \tau$  then  $\sum_{(i, j) \in \sigma} c((i, j)) < \sum_{(i, j) \in \tau} c((i, j))$ , and  $\sigma(\omega) > \tau(\omega)$ .*

*Proof.* By the theorems (1.15) and (1.16) in Macdonald [12, p.9] it suffices to prove the inequality for  $\tau = \sigma + \varepsilon(i) - \varepsilon(j)$  with  $i < j$  (this is a “raising operator”). Then  $\sum_{(i, j) \in \tau} c((i, j)) = \sum_{(i, j) \in \sigma} c((i, j)) + (\sigma_i - \sigma_j) + (j + 1 - i)$ .  $\square$

Recall the singular polynomials  $\zeta_\lambda^x$  associated to two-part partitions  $\tau = (\mu, N - \mu)$  with  $\lambda = (m^{N-\mu})$  and  $\gcd(m, \mu + 1) < \frac{\mu+1}{N-\mu}$ ; then  $\tau(\omega) = (\mu + 1)(N - \mu)$  and  $\deg \zeta_\lambda^x = m(N - \mu)$ . For  $\tau = (s(\mu + 1) + \mu, \mu^l, \rho)$  we find

$$\begin{aligned} \tau(\omega) &= (\mu + 1) \left( \rho(s + l + 1) + \frac{1}{2} \mu l(l + 2s + 1) \right) \\ &= \frac{\mu + 1}{m} |\Lambda(\mu, s, l, \rho, m)|. \end{aligned}$$

**Theorem 6.** *For  $\lambda = \Lambda(\mu, s, l, \rho, m)$  and  $\kappa = -\frac{m}{\mu+1}$  the singular polynomial  $\zeta_\lambda^x$  on  $\mathbb{R}^N$  is of isotype  $\tau = (s(\mu + 1) + \mu, \mu^l, \rho)$  ( $|\tau| = N$ ).*

*Proof.* For any  $\zeta_\sigma^x$  with  $\sigma \in \mathbb{N}_0^N$  if  $\sigma_i = \sigma_{i+1}$  for some  $i$  then  $(i, i + 1) \zeta_\sigma^x = \zeta_\sigma^x$ . Thus  $\zeta_\lambda^x$  is invariant under  $S_{[1, \rho]} \times \prod_{j=1}^l S_{[\rho+(j-1)\mu+1, \rho+j\mu]} \times S_{[\rho+l\mu+1, N]}$ , and this group is conjugate to  $S_\tau$  (the direct product  $\prod_i S_{\tau_i}$ ). Thus  $E = \text{span}_{\mathbb{Q}} \{w \zeta_\lambda^x : w \in S_N\}$  is isomorphic to a submodule of the representation of  $S_N$  induced up from  $1_{S_\tau}$ , the identity representation of  $S_\tau$ . By a classical theorem (see Macdonald [12, p.115]) this decomposes as a direct sum with one component of isotype  $\tau$  and all other components of isotypes  $\sigma$  with  $\sigma \succ \tau$ . Any  $f \in E$  is singular and  $f$  can not be of isotype  $\sigma \succ \tau$  because  $\deg f = |\lambda| = \frac{m}{\mu+1} \tau(\omega) > \frac{m}{\mu+1} \sigma(\omega)$  by the Lemma.  $\square$

The same method proves the following.

**Proposition 8.** *For  $\frac{N}{2} \leq \mu < N$ ,  $\gcd(m, \mu + 1) < \frac{\mu+1}{N-\mu}$ , and  $\lambda = (m^{N-\mu})$  the singular polynomial  $\zeta_\lambda^x$  on  $\mathbb{R}^N$  for  $\kappa = -\frac{m}{\mu+1}$  is of isotype  $(\mu, N - \mu)$ .*

We turn to the application of Murphy’s results. For any given isotype he determined the eigenvalues and eigenvectors of the commuting operators

$\left\{ \sum_{j=1}^{i-1} (i, j) : 2 \leq i \leq N \right\}$  (Jucys-Murphy elements). However the results have to be read in reverse in a certain sense.

**Proposition 9.** *Suppose  $f$  is a singular polynomial for  $\kappa = \kappa_0 \in \mathbb{Q}$  and  $1 \leq i \leq N$ , then  $\mathcal{U}_i f = f + \kappa_0 \sum_{j=i+1}^N (i, j) f$ .*

*Proof.* We have the commutation  $\mathcal{D}_i(x_i f) = x_i \mathcal{D}_i f + f + \kappa \sum_{j \neq i} (i, j) f$ . Now set  $\kappa = \kappa_0$  and note that  $\mathcal{U}_i f = \mathcal{D}_i(x_i f) - \kappa \sum_{j < i} (i, j) f$ .  $\square$

Denote the Murphy elements  $\omega_i = \sum_{j=N-i+2}^N (N+1-i, j)$  for  $2 \leq i \leq N$  and let  $\omega_1 = 0$  (as a transformation); then  $\mathcal{U}_i f = f + \kappa_0 \omega_{N+1-i} f$  for singular polynomials. Suppose that  $\zeta_\alpha^x$  is singular for  $\kappa = \kappa_0$  and  $\alpha = w\lambda$ , some  $w \in S_N$  (recalling that  $\mathcal{U}_i \zeta_\alpha^x = ((N - r(\alpha, i)) \kappa + \alpha_i + 1) \zeta_\alpha^x$ ), then  $\omega_{N+1-i} \zeta_\alpha^x = \left(N - r(\alpha, i) + \frac{\alpha_i}{\kappa_0}\right) \zeta_\alpha^x$ . A *standard Young tableau* (SYT) of shape  $\tau$  is a one-to-one assignment of the numbers  $\{1, \dots, N\}$  to the nodes of the Ferrers diagram so that the entries increase in each row and in each column. Let  $\eta_i(T)$  be the content of the node containing the value  $i$ ,  $1 \leq i \leq N$ . Murphy constructed a basis  $\{f_T : T \text{ is an SYT of shape } \tau\}$  for the irreducible representation of isotype  $\tau$  and  $\omega_i f_T = \eta_i(T) f_T$  for each  $i$  and  $T$ . There is an order on SYT's of given shape (for details see [13, p.288]) and the maximum SYT in this order, denoted by  $T_0$ , is produced by entering the numbers  $1, 2, \dots, N$  row by row (the first row is  $1, \dots, \tau_1$ , the second is  $\tau_1 + 1, \dots, \tau_1 + \tau_2$  and so forth).

**Definition 8.** Suppose  $T$  is an SYT of shape  $\tau$ , with  $\tau \in \mathbb{N}_0^{N,P}$  and  $|\tau| = N$ , then let  $rw(i, T)$ ,  $cm(i, T)$  denote the row and column respectively of the node of  $T$  containing  $i$ , for  $1 \leq i \leq N$ . Let  $t(i, \tau)$  (or  $t(i)$ ) =  $(rw(i, T_0), cm(i, T_0))$ , considered as a labeling of the nodes in the diagram of  $\tau$ .

In this notation  $\eta_i(T) = cm(i, T) - rw(i, T)$ .

**Proposition 10.** Let  $\lambda = \Lambda(\mu, s, l, \rho, m)$  (hypotheses as in Definition 6) then  $N - k + \frac{\lambda_k}{\kappa_0} = c(t(N+1-k)) = \eta_{N+1-k}(T_0)$  for  $1 \leq k \leq N$

*Proof.* Let  $c_k = N - k + \frac{\lambda_k}{\kappa_0} = N - k - (\mu + 1) \frac{\lambda_k}{m}$  for  $1 \leq k \leq N$ , then for  $k = \rho + 1 - j$  and  $1 \leq j \leq \rho$  we have  $c_k = j - (l + 2)$ , for  $k = \rho + (l + 1 - i)\mu + 1 - j$  with  $1 \leq i \leq l$  and  $1 \leq j \leq \mu$  we have  $c_k = j - (i + 1)$ , and finally for  $k = N + 1 - j$  with  $1 \leq j \leq N - \ell(\lambda) = s(\mu + 1) + \mu = \tau_1$  we have  $c_k = j - 1$ . Thus  $c_k = c(t(N+1-k))$ .  $\square$

For a partition  $\lambda \in \mathbb{N}_0^{N,P}$  say that  $w \in S_N$  is  $\lambda$ -rank-preserving if  $\lambda_i = \lambda_{i+1}$  implies  $w(i) < w(i+1)$  for  $1 \leq i < N$ . In general  $(w\lambda)_{w(i)} = \lambda_i$ , so this property implies  $r(w\lambda, i) = r(\lambda, w^{-1}(i)) = w^{-1}(i)$  for  $1 \leq i \leq N$  and

$$\mathcal{U}_i \zeta_{w\lambda}^x = ((N - w^{-1}(i)) \kappa + \lambda_{w^{-1}(i)} + 1) \zeta_{w\lambda}^x$$

for generic  $\kappa$ . In particular, if  $\lambda = \Lambda(\mu, s, l, \rho, m)$ ,  $w$  is  $\lambda$ -rank-preserving and  $\zeta_{w\lambda}^x$  is singular (for  $\kappa = \kappa_0$ ) then

$$\omega_{N+1-i} \zeta_{w\lambda}^x = c(t(N+1-w^{-1}(i))) \zeta_{w\lambda}^x.$$

Let  $w_0$  be the “reversing” (longest) element of  $S_N$ , that is  $w_0(i) = N + 1 - i$  for  $1 \leq i \leq N$  (note  $w_0^{-1} = w_0$ ). Thus  $\omega_i \zeta_{w\lambda}^x = c(t(w_0 w w_0(i))) \zeta_{w\lambda}^x$ . On the other hand suppose  $u \in S_N$  and the action of  $u$  on  $T_0$  produces an SYT denoted by  $T$  ( $u$  acts on the entries of  $T_0$ ), then the node  $t(i)$  contains  $u(i)$ . Thus  $\omega_i f_T = \eta_i(T) f_T = c(t(u^{-1}(i))) f_T$  and  $f_T$  has the same respective eigenvalues for  $\{\omega_i : 1 \leq i \leq N\}$  as  $\zeta_{w\lambda}^x$  for  $w = w_0^{-1} u w_0$  provided that  $w$  is  $\lambda$ -rank-preserving. But this is a consequence of  $T$  being an SYT ( $\lambda_i = \lambda_{i+1}$  implies  $N - i$  and  $N + 1 - i$  are in the same row of  $T_0$  thus  $u(N - i)$  and  $u(N - i + 1)$  are in the same row of  $T$ , with  $u(N - i) < u(N - i + 1)$ , that is,  $w(i) < w(i + 1)$ ). Further it is easy to

describe  $w\lambda$  corresponding to a given SYT  $T$ : let  $\gamma_1 = 0$  and  $\gamma_j = m(s + j - 1)$  for  $2 \leq j \leq l+2$ , then for any  $i$  (with  $1 \leq i \leq N$ ) let  $j = rw(i, T)$  and  $(w\lambda)_{N+1-i} = \gamma_j$ . This shows that the possible  $w\lambda$  corresponding to SYT's are exactly the reverse lattice permutations of  $\lambda$ . A *reverse lattice permutation*  $w\lambda$  of  $\lambda$  is defined by the property that every right substring  $(w\lambda)_{N+1-j}(w\lambda)_{N+2-j} \dots (w\lambda)_N$  (for  $1 \leq j \leq N$ ) has at least as many entries of  $\gamma_i$  as of  $\gamma_{i+1}$  for each  $i$ . The set of corresponding  $w \in S_N$  serves as an index set, namely  $E_\tau = \{w \in S_N : w_0 w w_0 T_0 \text{ is an SYT}\}$ .

We show that  $\{\zeta_{w\lambda}^x : w \in E_\tau\}$  is a basis for the  $S_N$ -module (isotype  $\tau$ ) generated by  $\zeta_\lambda^x$ .

**Theorem 7.** *Let  $w \in E_\tau$  then  $\zeta_{w\lambda}^x$  is singular (for  $\kappa = \kappa_0$  on  $\mathbb{R}^N$ ) and  $\{\zeta_{w\lambda}^x : w \in E_\tau\}$  is a basis for  $\text{span}_{\mathbb{Q}}\{w\zeta_\lambda^x : w \in S_N\}$ , on which  $S_N$  acts by Young's seminormal representation, where  $\zeta_{w\lambda}^x$  corresponds to  $f_T$  with  $T = w_0 w w_0 T_0$ .*

*Proof.* By Proposition 1 if  $\zeta_{w\lambda}^x$  has no pole at  $\kappa_0$ , for some  $w \in S_N$ , and  $a = \kappa(\kappa(r(w\lambda, i+1) - r(w\lambda, i)) + \lambda_{w^{-1}(i)} - \lambda_{w^{-1}(i+1)})^{-1}$  does not evaluate to  $\pm 1$  at  $\kappa = \kappa_0$ , for some  $i$  with  $\lambda_{w^{-1}(i)} > \lambda_{w^{-1}(i+1)}$  then  $\zeta_{(i,i+1)w\lambda}^x$  does not have a pole at  $\kappa_0$  (the formula is  $(i, i+1)\zeta_{w\lambda}^x = a\zeta_{w\lambda}^x + (1-a^2)\zeta_{(i,i+1)w\lambda}^x$ ). By Proposition 10  $a = (c(t(N+1-w^{-1}(i))) - c(t(N+1-w^{-1}(i+1))))^{-1}$  at  $\kappa = \kappa_0$ . Each SYT  $T$  of shape  $\tau$  is the result of a (finite) sequence  $\{(i_j, i_j+1) : 1 \leq j \leq n\}$  of adjacent transpositions applied to  $T_0$ , such that if  $T_j = (i_j, i_j+1)T_{j-1}$  then  $rw(i_j, T_{j-1}) < rw(i_j+1, T_{j-1})$ . This also implies that  $T_j$  is lower in the order on tableaux as used in [13].

For any SYT  $T$  there are four possibilities for the locations of  $i, i+1$  and Murphy [13, p.292] derived the expansion of  $(i, i+1)f_T$  in each case: if  $rw(i, T) = rw(i+1, T)$  then  $(i, i+1)f_T = f_T$ , if  $cm(i, T) = cm(i+1, T)$  then  $(i, i+1)f_T = -f_T$  and if  $rw(i, T) < rw(i+1, T)$  then  $(i, i+1)f_T = af_T + (1-a^2)f_{(i,i+1)T}$  where  $a = (\eta_i(T) - \eta_{i+1}(T))^{-1}$  (the fourth case,  $rw(i, T) > rw(i+1, T)$  follows from the previous by interchanging  $T$  and  $(i, i+1)T$ ; also  $rw(i, T) < rw(i+1, T)$  implies  $cm(i, T) > cm(i+1, T)$  thus  $0 < a \leq \frac{1}{2}$ ). As remarked before, if  $w\lambda$  corresponds to an SYT  $T$  with  $rw(i, T) < rw(i+1, T)$  then  $(w\lambda)_{N+1-i} < (w\lambda)_{N-i}$ . Let  $\beta = (N-i, N-i+1)w\lambda$ , then  $\zeta_\beta^x = (1-a^2)^{-1}((N-i, N-i+1)\zeta_{w\lambda}^x - a\zeta_{w\lambda}^x)$  with the same  $a$  that appears in the expression for  $f_{(i,i+1)T}$  in terms of  $f_T$  (note  $w_0(i, i+1)w_0 = (N-i, N-i+1)$ ). Since  $f_{T_0}$  has the same eigenvalues for  $\{\omega_i\}$  as  $\zeta_\lambda^x$  this argument used inductively (on the number of adjacent transpositions linking  $T_0$  to  $T$ ) shows that  $\{\zeta_{w\lambda}^x : w \in E_\tau\}$  transforms according to the seminormal representation, for the isotype  $\tau$ . Again suppose  $\zeta_{w\lambda}^x$  corresponds to the SYT  $T$  (that is  $T = w_0 w w_0 T_0$ ); if  $rw(i, T) = rw(i+1, T)$  then  $(w\lambda)_{N+1-i} = (w\lambda)_{N-i}$  and  $\zeta_{w\lambda}^x$  is invariant under  $(N-i, N-i+1)$ , while if  $cm(i, T) = cm(i+1, T)$  then  $\eta_i(T) - \eta_{i+1}(T) = 1$  and the equation  $(N-i, N-i+1)\zeta_{w\lambda}^x = -\zeta_{w\lambda}^x$  is a consequence of the fact that  $S_N$  acts on the basis  $\{\zeta_{w\lambda}^x : w \in E_\tau\}$  just as on  $\{f_T\}$ .  $\square$

The concept of reverse lattice permutations of  $\lambda$  provides a concise labeling of the singular polynomials of isotype  $\tau$ .

## 5. CONCLUSION

Here is a description of how to find the isotype  $\tau$  and label  $\lambda$  for the singular value  $\kappa = -\frac{m}{n}$ , given a pair  $(m, n)$  with  $2 \leq n \leq N, m \geq 1$  and  $\frac{m}{n} \notin \mathbb{N}$ . Let  $d = \gcd(m, n)$ ,  $m_1 = \frac{m}{d}$ ,  $n_1 = \frac{n}{d}$  (by hypothesis  $n_1 \geq 2$ ), then let

$l = \left\lceil \frac{N+1-n}{n_1-1} \right\rceil - 1$  (the ceiling function),  $\rho = (N+1-n) - l(n_1-1)$ . If  $l = 0$  then  $\tau = (n-1, N+1-n)$  and  $\lambda = (m^{N+1-n}, 0^{n-1})$ . If  $l \geq 1$  then  $\tau = (n-1, (n_1-1)^l, \rho)$ ,  $\mu = n_1-1$ ,  $s = d-1$  and  $\lambda = \Lambda(n_1-1, d-1, l, \rho, m_1)$ , that is,  $\lambda = ((m+lm_1)^\rho, (m+(l-1)m_1)^\mu, \dots, m^\mu, 0^{n-1})$ . Note that the first part of  $\tau$  is always  $n-1$  (and  $\lambda$  ends in  $n-1$  zeros).

The rational Cherednik algebra  $\mathbf{A}$  was investigated by Berest, Chmutova, Etingof, Ginzburg, Guay, Opdam and Rouquier in a series of papers [1],[2],[3],[9],[10]. Here we consider the faithful representation of  $\mathbf{A}$  as the algebra generated by  $\{\mathcal{D}_i, x_i : 1 \leq i \leq N\} \cup S_N$  of operators on polynomials on  $\mathbb{R}^N$  (where  $x_i$  denotes the multiplication operator). Suppose  $\kappa = \kappa_0$  is a singular value and  $\tau, \lambda$  are defined as in the previous section and used in Theorem 7, and let  $M_\tau = \text{span}_{\mathbb{P}} \{\zeta_{w\lambda}^x : w \in E_\tau\}$  where  $\mathbb{P}$  denotes  $\mathbb{Q}[x_1, \dots, x_N]$ , the polynomials on  $\mathbb{R}^N$ . Then  $M_\tau$  is a module for the Cherednik algebra  $\mathbf{A}$  specialized to  $\kappa = \kappa_0$ ; clearly  $M_\tau$  is closed under multiplication by polynomials and the action of  $S_N$ . It is closed under  $\{\mathcal{D}_i : 1 \leq i \leq N\}$ , indeed suppose  $p$  is a polynomial and  $g \in \text{span}_{\mathbb{Q}} \{\zeta_{w\lambda}^x : w \in E_\tau\}$  then by the product rule  $\mathcal{D}_i(pg) = p\mathcal{D}_i g + g \frac{\partial}{\partial x_i} p + \kappa \sum_{j \neq i} ((i, j)g) \frac{p(x) - (i, j)p(x)}{x_i - x_j} \in M_\tau$  when  $\kappa = \kappa_0$  and  $\mathcal{D}_i g = 0$ .

It is a plausible conjecture that we have found all the singular polynomials for  $S_N$  (perhaps to be settled in a later paper). The structure of  $\kappa_0$ -critical pairs (see Definition 7) may be worth further investigation, with a view to finding a general algorithm for their construction, and maybe a uniqueness result in the case that  $h(\lambda, \kappa+1)$  has a zero of multiplicity 1 at  $\kappa = \kappa_0$ . Such a result would simplify the argument used here.

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